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NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

A NONLINEAR WAVE SHOALING MODEL FOR ALONGSHORE VARYING BATHYMETRY

by

David M. Ruth

September 2001

Thesis Advisor:	Thomas H.C. Herbers
Second Reader:	Edward B. Thornton

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This thesis proposes an improvement to present near-shore wave prediction models. Using weakly dispersive Boussinesq theory, the shoaling of directionally spread surface gravity waves over a beach with gentle gradients in the cross-shore and alongshore directions is examined. Following Herbers and Burton (1997), the governing fluid flow equations are expanded to third order and depth-integrated over the water column. A resulting amplitude evolution equation for a spectrum of waves is derived, which is the main result of this paper. New terms in the higher order result include effects due to alongshore bottom slope, higher order cross-shore depth variations, and non-linear quartet interactions. The linear terms in this equation are verified by analytical methods using linear finite depth theory. Example computations for a monochromatic wave train over a plane beach quantify some of the improvements of this result over the lower order model. Opportunities for further development and verification of this result are proposed, and recommendations for application of the result in its present form are outlined.

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A NONLINEAR WAVE SHOALING MODEL FOR ALONGSHORE VARYING BATHYMETRY

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Lieutenant Commander, United States Navy
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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

This thesis proposes an improvement to present near-shore wave prediction models. Using weakly dispersive Boussinesq theory, the shoaling of directionally spread surface gravity waves over a beach with gentle gradients in the cross-shore and alongshore directions is examined. Following Herbers and Burton (1997), the governing fluid flow equations are expanded to third order and depth-integrated over the water column. A resulting amplitude evolution equation for a spectrum of waves is derived, which is the main result of this paper. New terms in the higher order result include effects due to alongshore bottom slope, higher order cross-shore depth variations, and non-linear quartet interactions. The linear terms in this equation are verified by analytical methods using linear finite depth theory. Example computations for a monochromatic wave train over a plane beach quantify some of the improvements of this result over the lower order model. Opportunities for further development and verification of this result are proposed, and recommendations for application of the result in its present form are outlined.

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I. INTRODUCTION

This thesis proposes an improvement to present near-shore wave prediction models. Accurate prediction of wave shoaling transformation is crucial for the successful planning of amphibious operations, providing a strong military motivation for this research. Additionally, the action of waves and wave-driven currents on a beach drives sediment transport and beach erosion, stimulating civilian interest in this field as well.

As ocean surface waves shoal from deep to shallow water, amplitudes increase, wavelengths decrease, and propagation directions refract toward normal incidence to the beach as a result of linear processes. Additionally, non-linear interactions can cause significant energy transfers to wave components with both higher and lower frequencies than the original waveform. In deep water ($\kappa h \gg 1$, where κ is the wave number and h is the water depth) and intermediate depths ($\kappa h = O(1)$), nonlinear effects on waves are usually evaluated with finite depth theory on the basis of Stokes perturbation expansion for small wave slopes ($a\kappa \ll 1$, where a is the wave amplitude) (e.g., *Phillips,* 1960; *Hasselmann,* 1962). Nonlinear interaction between two primary wave components excites secondary waves with frequencies and wave numbers that are the sum (or difference) of those of the primary waves. In deep and intermediate water depths, these "triad" interactions are nonresonant, and the resulting secondary wave amplitudes are small compared to the primary wave amplitudes (*Phillips*, 1960; *Herbers et al.,* 1992, 1994). Similarly, tertiary waves may be excited by non-linear interaction between three primary wave components, constituting "quartet" interactions. For certain combinations of wave numbers quartet interactions are resonant, causing a gradual exchange of energy between the wave components that is important to the evolution of wind-wave spectra over large distances (*Phillips,* 1960; *Hasselmann,* 1962; *Hasselmann et al.,* 1973).

Unlike the deep and intermediate water regimes, in shallow water ($\kappa h \ll 1$) the secondary response is strongly amplified due to near-resonance. The conventional approach to describe these interactions is the use of Boussinesq-type equations, which are based on the assumption that measures of nonlinearity, $\varepsilon\ (\equiv a/h)$, and dispersion $\sigma^2(\equiv(\kappa h)^2)$ are both small and of the same order. This approach reduces the three-dimensional fluid motion problem into a two-dimensional one by introducing a polynomial approximation of the vertical structure of the flow field, while accounting for weak non-hydrostatic effects due to the vertical fluid acceleration (see *Madsen & Schäffer*, 1998, for a comprehensive review). Boussinesq's original work (1872) was restricted to a horizontal bottom; since then that work has been extended to an uneven bottom in one dimension (*Mei & LeMéhauté*, 1966), and later two dimensions (*Peregrine*, 1967).

Herbers and Burton (1997, referred to as HB in the following) formulate shoaling evolution equations for the spectral amplitudes and phases of a directionally spread wave field propagating over a shallow, gently sloping beach with straight and parallel depth contours that are accurate to the lowest order (ε) in nonlinearity and dispersion (see also *Freilich & Guza*, 1984; and *Elgar & Guza*, 1985). Limitations of HB include: 1) neglect of alongshore depth variation, 2) restriction to gently sloping beaches, 3) restriction to small oblique wave angles, 4) a low order approximation of dispersion effects which limits the model to very shallow water (nominally depths less than 10m), and 5) neglect of nonlinear interactions between more than three wave components. In this paper, we propose improvements that address the limitations mentioned above by carrying out the HB derivation to one higher order (ε^2) in nonlinearity, dispersion, and bottom slope, and allowing for weak alongshore depth variations. This higher order model still assumes a gently sloping beach and small oblique angles, but improves the accuracy of the model for finite bottom slopes and angles. Dispersion, directional spreading, and strength of triad interactions are all represented to one higher order, as well as the effects of the cross-shore bottom slope. New effects introduced at the lowest order are alongshore depth variation and nonlinear quartet interactions.

An overview of the technique used to obtain this result is as follows: The first order horizontal flow and pressure is shown to be depth independent. The continuity equation is then depth-integrated to obtain a lowest order expression for the vertical flow. This vertical flow solution is substituted into the irrotationality condition and the vertical momentum equation, which are then depth-integrated to obtain the next higher order expressions for the horizontal flow and pressure, respectively. This process is repeated until third order expressions for the horizontal flow and pressure are obtained. Following Freilich and Guza (1984) and HB, an assumed first order spectral wave field is substituted in the governing equation for the depth averaged velocity potential, and the solvability condition yields a coupled set of first order equations for the evolution of the spectral amplitudes.

A significant challenge to arriving at such higher order solutions is that the complexity of the polynomials at each level of the derivation grows significantly as higher orders of non-linearity and dispersion are considered. Madsen & Schäffer (1998), Agnon *et al.* (1999), and Gobbi *et al.* (2000) derive a variety of Boussinesq-type equations to high orders of accuracy, from which this complexity is quite evident. Although equations derived in these earlier studies are more general than those that follow in this thesis, they are not cast in a spectral form and require prohibitively expensive full time-space domain solution techniques.

This thesis is organized as follows. In Chapter II the formulation of governing fluid flow equations are expanded to third order and integrated over the water column. A coupled set of amplitude evolution equations for a spectrum of waves, the main result of this paper, is derived in Chapter III. In Chapter IV, linear terms in these equations are verified by analytical methods, and example computations for a plane beach quantify some of the improvements of this model over the lower order HB model. Finally, Chapter V summarizes these results, and identifies opportunities for further research.

II. THIRD ORDER GOVERNING EQUATIONS

A. ASSUMPTIONS

The equations of motion and the continuity equation may be written in nondimensionalized form (neglecting viscous effects):

$$\begin{gathered}\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\bullet\nabla)\,\mathbf{u}+w\frac{\partial \mathbf{u}}{\partial z}+\nabla p=0,\\ \frac{\partial w}{\partial t}+(\mathbf{u}\bullet\nabla)\,w+w\frac{\partial w}{\partial z}+\frac{\partial p}{\partial z}+1=0,\\ \nabla\bullet\mathbf{u}+\frac{\partial w}{\partial z}=0,\end{gathered}\tag{1}$$

where *t* is time, ∇ denotes the two-dimensional gradient operator ($\partial/\partial x,\partial/\partial y$), **u** is the vector (*u,v*) of the horizontal (*x,y*) velocity components, *w* is the vertical (*z*) velocity component, and *p* is pressure. The *x*-axis points onshore, *y* points alongshore, and *z* points upward with $z=0$ corresponding to the mean sea surface. The surface and bottom are defined by $\eta(x,y,t)$ and $z=-h(x,y)$, as shown in Figure 2.1.

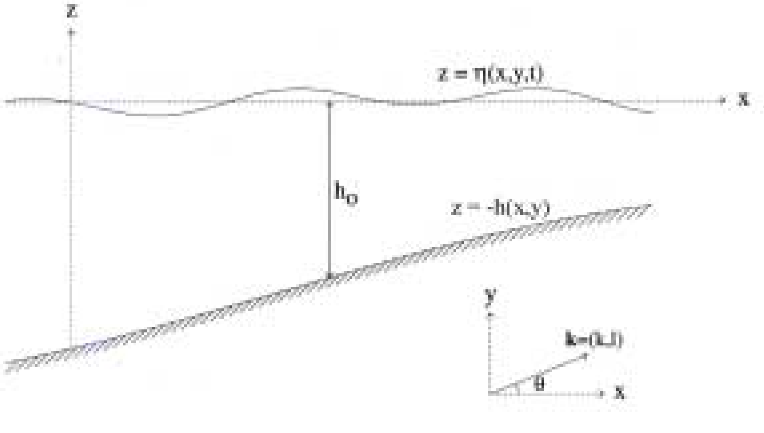

Figure 2.1. Definitional sketch of variables and coordinate frame. Directionally spread waves propagate over a beach with gentle cross-shore and alongshore depth gradients. The variable θ denotes the wave propagation direction, **k** is the wavenumber vector, and h_0 is a representative water depth.

All variables in equation (1) are normalized using gravity *g*, density of seawater ρ, and a representative water depth h_0. Irrotational flow is assumed, so that

$$\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}, \qquad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}. \tag{2}$$

The surface and bottom boundary conditions are given by

$$\begin{aligned} w&=\frac{\partial \eta}{\partial t}+\mathbf{u}\bullet\nabla\eta, \qquad && z=\eta, \\ p&=0, && z=\eta, \\ w&=-\mathbf{u}\bullet\nabla h, && z=-h. \end{aligned} \tag{3}$$

The variables η, *u, v, w,* and *p* are expanded in terms of the non-linearity parameter ε, defined to be the ratio of wave amplitude, *a*, to water depth, *h*. In shallow water, *v* and *w* are both $O(\kappa h)$ smaller than *u* (where κ is the wavenumber), and thus the perturbation expansions of the scaled variables are given by

$$\begin{aligned} \eta &= \varepsilon\eta_1+\varepsilon^2\eta_2+\varepsilon^3\eta_3+..., \\ u &= \varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+..., \\ v &= \sigma(\varepsilon v_1+\varepsilon^2 v_2+\varepsilon^3 v_3+...), \\ w &= \sigma(\varepsilon w_1+\varepsilon^2 w_2+\varepsilon^3 w_3+...), \\ p &= -z+\varepsilon p_1+\varepsilon^2 p_2+\varepsilon^3 p_3+..., \end{aligned} \tag{4}$$

where $\sigma\equiv\kappa h$ is a measure of dispersion. The independent variables are then scaled

$$\frac{\partial}{\partial x}=\sigma\frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial y}=\sigma^2\frac{\partial}{\partial y'}, \qquad \frac{\partial}{\partial z}=\frac{\partial}{\partial z'}, \qquad \frac{\partial}{\partial t}=\sigma\frac{\partial}{\partial t'}, \tag{5}$$

so that the order of the terms appears explicitly when equations (4) and (5) are substituted into the governing equations. In the classic Boussinesq approximation, dispersion σ^2 and nonlinearity ε are assumed to be of the same order. The water depth is taken to be a function of the slow variables $(\tilde{x},\tilde{y})$ such that $\dfrac{\partial h}{\partial x'}=\beta\dfrac{\partial h}{\partial \tilde{x}}$ and $\dfrac{\partial h}{\partial y'}=\beta\dfrac{\partial h}{\partial \tilde{y}}$, where $\beta\equiv\nabla h/\kappa h$ is assumed to be $O(\varepsilon)$. In this approximation, alongshore depth variations are $O(\kappa h)$ smaller than cross-shore variations.

B. VERTICAL STRUCTURE OF THE FLOW AND PRESSURE FIELD

In this section the irrotationality conditions, vertical momentum equation, and continuity equation are integrated over the water column at successive orders of ε to determine the vertical structure of the flow and pressure field accurate to $O(\varepsilon^3)$. First, we expand the irrotationality conditions (2) and vertical momentum equation (1b) using (4b-e) and (5) to $O(\varepsilon^3)$:

$$\frac{\partial u_1}{\partial z'}+\varepsilon\left(\frac{\partial u_2}{\partial z'}-\frac{\partial w_1}{\partial x'}\right)+\varepsilon^2\left(\frac{\partial u_3}{\partial z'}-\frac{\partial w_2}{\partial x'}\right)=O(\varepsilon^3), \tag{6}$$

$$\frac{\partial v_1}{\partial z'}+\varepsilon\left(\frac{\partial v_2}{\partial z'}-\frac{\partial w_1}{\partial y'}\right)+\varepsilon^2\left(\frac{\partial v_3}{\partial z'}-\frac{\partial w_2}{\partial y'}\right)=O(\varepsilon^3), \tag{7}$$

$$\left(\frac{\partial u_1}{\partial y'}-\frac{\partial v_1}{\partial x'}\right)+\varepsilon\left(\frac{\partial u_2}{\partial y'}-\frac{\partial v_2}{\partial x'}\right)+\varepsilon^2\left(\frac{\partial u_3}{\partial y'}-\frac{\partial v_3}{\partial x'}\right)=O(\varepsilon^3), \tag{8}$$

$$\frac{\partial p_1}{\partial z'}+\varepsilon\left(\frac{\partial p_2}{\partial z'}+\frac{\partial w_1}{\partial t'}\right)+\varepsilon^2\left(\frac{\partial p_3}{\partial z'}+\frac{\partial w_2}{\partial t'}+u_1\frac{\partial w_1}{\partial x'}+w_1\frac{\partial w_1}{\partial z'}\right)=O(\varepsilon^3). \tag{9}$$

Note that u_1, v_1, and p_1 are independent of z' to $O(\varepsilon)$. Substitution of the dynamic surface boundary condition (3b) in (4e) yields

$$p_1=\eta_1. \tag{10}$$

Next we expand the continuity equation (1c) and the kinematic bottom boundary condition (3c) and collect terms of like order,

$$\frac{\partial u_1}{\partial x'}+\frac{\partial w_1}{\partial z'}=0, \quad w_1=0 \text{ at } z'=-h, \tag{11}$$

$$\frac{\partial u_2}{\partial x'}+\frac{\partial v_1}{\partial y'}+\frac{\partial w_2}{\partial z'}=0, \quad w_2=-u_1\frac{\partial h}{\partial \tilde{x}} \text{ at } z'=-h. \tag{12}$$

At the lowest order, equation (11) integrates to

$$w_1=-\left(z'+h\right)\frac{\partial u_1}{\partial x'}. \tag{13}$$

Substitution of w_1 and p_1 into (6), (7), and (9) and integration over the water column yields the vertical distribution of the second-order velocity and pressure field

$$u_2=U_2-\left(\frac{z'^2}{2}+hz'\right)\frac{\partial^2 u_1}{\partial x'^2}, \tag{14}$$

$$v_2=V_2-\left(\frac{z'^2}{2}+hz'\right)\frac{\partial^2 u_1}{\partial x'\partial y'}, \tag{15}$$

$$p_2=P_2-\left(\frac{z'^2}{2}+hz'\right)\frac{\partial^2 u_1}{\partial x'\partial t'}, \tag{16}$$

where U_2, V_2, and P_2 are integration constants.

The constant P_2 follows from substituting p_1 and p_2 in (4) and evaluating the dynamic surface boundary condition (3b) to $O\left(\varepsilon^2\right)$

$$P_2=\eta_2. \tag{17}$$

The remaining constants U_2 and V_2 can be expressed in terms of the second-order depth-averaged flow variables $\overline{u_2}$ and $\overline{v_2}$ (equations (73) and (76) in Appendix A). With these results, the second order wave motion can be written as

$$u_2 = \overline{u_2} - \left(\frac{z'^2}{2} + hz' + \frac{h^2}{3} \right) \frac{\partial^2 u_1}{\partial x'^2}, \tag{18}$$

$$v_2 = \overline{v_2} - \left(\frac{z'^2}{2} + hz' + \frac{h^2}{3} \right) \frac{\partial^2 u_1}{\partial x' \partial y'}, \tag{19}$$

$$p_2 = \eta_2 - \left(\frac{z'^2}{2} + hz' \right) \frac{\partial^2 u_1}{\partial x' \partial t'}. \tag{20}$$

Having determined the second order pressure and horizontal flow variables, we may proceed as in the case of w_1 and integrate (12) to determine w_2:

$$w_2 = \left(\frac{z'^3}{6} + \frac{hz'^2}{2} + \frac{h^2 z'}{3} \right) \frac{\partial^3 u_1}{\partial x'^3} - \left(z' + h \right) \left(\frac{\partial \overline{u_2}}{\partial x'} + \frac{\partial v_1}{\partial y'} \right) - u_1 \frac{\partial h}{\partial \tilde{x}}. \tag{21}$$

Next we evaluate u_3 by substituting w_1, u_2, v_2, and w_2 into equation (6) and integrating the $O\left(\varepsilon^2\right)$ terms with respect to z':

$$\begin{aligned} u_3 = U_3 &- \left(\frac{z'^2}{2} + hz' \right) \left(\frac{\partial^2 \overline{u_2}}{\partial x'^2} + \frac{\partial^2 v_1}{\partial x' \partial y'} \right) + \left(\frac{z'^4}{24} + \frac{hz'^3}{6} + \frac{h^2 z'^2}{6} \right) \frac{\partial^4 u_1}{\partial x'^4} \\ &- 2z \frac{\partial h}{\partial \tilde{x}} \frac{\partial u_1}{\partial x'}. \end{aligned} \tag{22}$$

Expressing the integration constant U_3 in terms of the depth-averaged third order flow $\overline{u_3}$ using equation (74) (Appendix A), we obtain

$$
\begin{aligned}
u_3 = \overline{u_3} &+ \left(\frac{z'^4}{24} + \frac{hz'^3}{6} + \frac{h^2 z'^2}{6} - \frac{h^4}{45}\right)\frac{\partial^4 u_1}{\partial x'^4} \\
&- \left(\frac{z'^2}{2} + hz' + \frac{h^2}{3}\right)\left(\frac{\partial^2 \overline{u_2}}{\partial x'^2} + \frac{\partial^2 v_1}{\partial x' dy'}\right) \\
&- \left(2z' + h\right)\frac{\partial h}{\partial \tilde{x}}\frac{\partial u_1}{\partial x'} + \frac{h\eta_1}{3}\frac{\partial^2 u_1}{\partial x'^2}.
\end{aligned}
\tag{23}
$$

Similarly for v_3,

$$
\begin{aligned}
v_3 = V_3 &- \left(\frac{z'^2}{2} + hz'\right)\left(\frac{\partial^2 \overline{u_2}}{\partial x' \partial y'} + \frac{\partial^2 v_1}{\partial y'^2}\right) + \left(\frac{z'^4}{24} + \frac{hz'^3}{6} + \frac{h^2 z'^2}{6}\right)\frac{\partial^4 u_1}{\partial x'^3 \partial y'} \\
&- z\left(\frac{\partial h}{\partial \tilde{y}}\frac{\partial u_1}{\partial x'} + \frac{\partial h}{\partial \tilde{x}}\frac{\partial u_1}{\partial y'}\right),
\end{aligned}
\tag{24}
$$

and substituting for V_3 using equation (77) (Appendix A) yields

$$
\begin{aligned}
v_3 = \overline{v_3} &+ \left(\frac{z'^4}{24} + \frac{hz'^3}{6} + \frac{h^2 z'^2}{6} - \frac{h^4}{45}\right)\frac{\partial^4 u_1}{\partial x'^3 dy'} - \left(\frac{z'^2}{2} + hz' + \frac{h^2}{3}\right)\left(\frac{\partial^2 \overline{u_2}}{\partial x' \partial y'} + \frac{\partial^2 v_1}{dy'^2}\right) \\
&- \left(z' + \frac{h}{2}\right)\left(\frac{\partial u_1}{\partial y'}\frac{\partial h}{\partial \tilde{x}} + \frac{\partial u_1}{\partial x'}\frac{\partial h}{\partial \tilde{y}}\right) + \frac{\eta_1 h}{3}\frac{\partial^2 u_1}{\partial x' \partial y'}.
\end{aligned}
\tag{25}
$$

Finally, p_3 is determined by substituting p_1, p_2, w_1, and w_2 into the vertical momentum equation (1b) and depth-integrating the $O\left(\varepsilon^2\right)$ terms:

$$
\begin{aligned}
p_3 = P_3 &+ \left(\frac{z'^2}{2} + hz'\right)\left(\frac{\partial^2 \overline{u_2}}{\partial x' \partial t'} + \frac{\partial^2 v_1}{\partial y' \partial t'} + u_1\frac{\partial^2 u_1}{\partial x'^2} - \left(\frac{\partial u_1}{\partial x'}\right)^2\right) \\
&- \left(\frac{z'^4}{24} + \frac{hz'^3}{6} + \frac{h^2 z'^2}{6}\right)\frac{\partial^4 u_1}{\partial x'^3 \partial t'} + z'\frac{\partial h}{\partial \tilde{x}}\frac{\partial u_1}{\partial t'}.
\end{aligned}
\tag{26}
$$

From equation (3b) it follows that

$$\begin{aligned} p\big|_{z=\eta} &= -\varepsilon\left(\eta_1 + \varepsilon\eta_2 + \varepsilon^2\eta_3 + O(\varepsilon^3)\right) + \varepsilon\eta_1 \\ &+\varepsilon^2\left(\eta_2 + \varepsilon\eta_1 h\frac{\partial^2 u_1}{\partial x'\partial t'} + O(\varepsilon^2)\right) + \varepsilon^3\left(P_3 + O(\varepsilon)\right) + O(\varepsilon^4) = 0. \end{aligned} \tag{27}$$

Hence,

$$P_3 = \eta_3 - \eta_1 h\frac{\partial^2 u_1}{\partial x'\partial t'} \tag{28}$$

and

$$\begin{aligned} p_3 = \eta_3 &+ \left(\frac{z'^2}{2} + hz'\right)\left(\frac{\partial^2 \overline{u_2}}{\partial x'\partial t'} + \frac{\partial^2 v_1}{\partial y'\partial t'} + u_1\frac{\partial^2 u_1}{\partial x'^2} - \left(\frac{\partial u_1}{\partial x'}\right)^2\right) \\ &-\left(\frac{z'^4}{24} + \frac{hz'^3}{6} + \frac{h^2 z'^2}{6}\right)\frac{\partial^4 u_1}{\partial x'^3\partial t'} + z'\frac{\partial h}{\partial \tilde{x}}\frac{\partial u_1}{\partial t'} - \eta_1 h\frac{\partial^2 u_1}{\partial x'\partial t'}. \end{aligned} \tag{29}$$

In summary, the flow and pressure fields are described by equations (10) and (13)-(29). Equations (10) and (13)-(20) are identical to those in HB. The higher order equations, (21)-(29), improve the HB results by expressing the vertical flow structure as a fourth order polynomial in depth, and by including nonlinear and bottom slope terms that are neglected at lower orders.

C. REDUCTION OF THE GOVERNING EQUATIONS

Here the velocity and pressure distributions derived in the previous section are substituted into the governing equations to obtain depth-integrated horizontal momentum and continuity equations. First, the perturbation expansions (4) and scaling relations (5) are substituted into the horizontal momentum equation (1a). Collecting terms of like order,

$$\begin{aligned}&\frac{\partial u_1}{\partial t'}+\frac{\partial p_1}{\partial x'}+\varepsilon\left(\frac{\partial u_2}{\partial t'}+u_1\frac{\partial u_1}{\partial x'}+\frac{\partial p_2}{\partial x'}\right)\\&\qquad+\varepsilon^2\left(\frac{\partial u_3}{\partial t'}+\frac{\partial\left(u_1u_2\right)}{\partial x'}+v_1\frac{\partial u_1}{\partial y'}+w_1\frac{\partial u_2}{\partial z'}+\frac{\partial p_3}{\partial x'}\right)=O(\varepsilon^3)\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial v_1}{\partial t'}+\frac{\partial p_1}{\partial y'}+\varepsilon\left(\frac{\partial v_2}{\partial t'}+u_1\frac{\partial v_1}{\partial x'}+\frac{\partial p_2}{\partial x'}\right)\\&\qquad+\varepsilon^2\left(\frac{\partial v_3}{\partial t'}+u_1\frac{\partial v_2}{\partial x'}+u_2\frac{\partial v_1}{\partial x'}+v_1\frac{\partial v_1}{\partial y'}+w_1\frac{\partial v_2}{\partial z'}+\frac{\partial p_3}{\partial y'}\right)=O(\varepsilon^3).\end{aligned}$$

Substituting w_1, u_2, v_2, p_2, u_3, v_3 and p_3 (equations (13), (18)-(20), (23), (25), (29)) yields after some algebraic manipulation

$$\begin{aligned}&\frac{\partial u_1}{\partial t'}+\frac{\partial \eta_1}{\partial x'}+\varepsilon\left(\frac{\partial \overline{u_2}}{\partial t'}+\frac{\partial \eta_2}{\partial x'}+u_1\frac{\partial u_1}{\partial x'}-\frac{h^2}{3}\frac{\partial^3 u_1}{\partial x'^2\partial t'}\right)\\&+\varepsilon^2\left(\frac{\partial \overline{u_3}}{\partial t'}+\frac{\partial \eta_3}{\partial x'}-\frac{h^2}{3}\left(\frac{\partial^3\overline{u_2}}{\partial x'^2\partial t'}+\frac{\partial^3 v_1}{\partial x' dy'\partial t'}\right)-\frac{h^4}{45}\frac{\partial^5 u_1}{\partial x'^4\partial t'}-h\frac{\partial h}{\partial \tilde{x}}\frac{\partial^2 u_1}{\partial x'\partial t'}\right.\\&\quad+\frac{h}{3}\frac{\partial}{\partial t'}\left(\eta_1\frac{\partial^2 u_1}{\partial x'^2}\right)+\frac{\partial}{\partial x'}\left(u_1\overline{u_2}\right)+v_1\frac{\partial u_1}{dy'}\\&\quad\left.+\frac{h^2}{3}\left(2\frac{\partial u_1}{\partial x'}\frac{\partial^2 u_1}{\partial x'^2}-u_1\frac{\partial^3 u_1}{\partial x'^3}\right)-h\frac{\partial}{\partial x'}\left(\eta_1\frac{\partial^2 u_1}{\partial x'\partial t'}\right)\right)=O(\varepsilon^3),\end{aligned}\tag{30}$$

$$\begin{aligned}&\frac{\partial v_1}{\partial t'}+\frac{\partial \eta_1}{\partial y'}+\varepsilon\left(\frac{\partial \overline{v_2}}{\partial t'}+\frac{\partial \eta_2}{\partial y'}+u_1\frac{\partial v_1}{\partial x'}-\frac{h^2}{3}\frac{\partial^3 u_1}{\partial x'\partial y'\partial t'}\right)\\&+\varepsilon^2\left(\frac{\partial \overline{v_3}}{\partial t'}+\frac{\partial \eta_3}{\partial y'}-\frac{h^2}{3}\left(\frac{\partial^3\overline{u_2}}{\partial x'\partial y'\partial t'}+\frac{\partial^3 v_1}{dy'^2\partial t'}\right)-\frac{h^4}{45}\frac{\partial^5 u_1}{\partial x'\partial y'\partial t'}-\frac{h}{2}\left(\frac{\partial h}{\partial \tilde{y}}\frac{\partial^2 u_1}{\partial x'\partial t'}+\frac{\partial h}{\partial \tilde{x}}\frac{\partial^2 u_1}{\partial y'\partial t'}\right)\right.\\&\quad+\frac{h}{3}\frac{\partial}{\partial t'}\left(\eta_1\frac{\partial^2 u_1}{\partial x'\partial y'}\right)+u_1\frac{\partial \overline{v_2}}{dx'}+\overline{u_2}\frac{\partial v_1}{dx'}+v_1\frac{\partial v_1}{dy'}\\&\quad\left.+h^2\frac{\partial u_1}{\partial x'}\frac{\partial^2 u_1}{\partial x'\partial y'}-\frac{h^2}{3}\frac{\partial}{\partial y'}\left(u_1\frac{\partial^2 u_1}{\partial x'^2}\right)-h\frac{\partial}{\partial y'}\left(\eta_1\frac{\partial^2 u_1}{\partial x'\partial t'}\right)\right)=O(\varepsilon^3).\end{aligned}\tag{31}$$

Finally, integrating the continuity equation (1c) over the entire water column, applying Leibniz Theorem, and the kinematic surface and bottom boundary conditions (3a,c),

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\int_{-h}^{\eta} u dz+\frac{\partial}{\partial y'}\int_{-h}^{\eta} v dz=0. \tag{32}$$

Substituting the expansions (4a-c) and (5) yields

$$\begin{aligned}&\frac{\partial}{\partial t'}\left(\varepsilon\eta_1+\varepsilon^2\eta_2+\varepsilon^3\eta_3+O(\varepsilon^4)\right)+\sigma\frac{\partial}{\partial x'}\int_{-h}^{\eta}\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+O(\varepsilon^4)\right)dz'\\&+\sigma^2\frac{\partial}{\partial y'}\int_{-h}^{\eta}\sigma\left(\varepsilon v_1+\varepsilon^2 v_2+O(\varepsilon^3)\right)dz'=0.\end{aligned} \tag{33}$$

Now evaluate the integrals for each of the flow variables using equations (4a), (18), (19), (23), and (25):

$$\int_{-h}^{\eta} u_1 dz'=\left(h+\varepsilon\eta_1+\varepsilon^2\eta_2\right)u_1+O(\varepsilon^3), \tag{34}$$

$$\int_{-h}^{\eta} u_2 dz'=\left(h+\varepsilon\eta_1\right)\overline{u_2}-\varepsilon\eta_1 h^2\frac{\partial^2 u_1}{\partial x'^2}+O(\varepsilon^2), \tag{35}$$

$$\int_{-h}^{\eta} u_3 dz'=h\overline{u_3}+\frac{\eta_1 h^2}{3}\frac{\partial^2 u_1}{\partial x'^2}+O\left(\varepsilon\right), \tag{36}$$

$$\int_{-h}^{\eta} v_1 dz'=\left(h+\varepsilon\eta_1\right)v_1+O(\varepsilon^2), \tag{37}$$

$$\int_{-h}^{\eta} v_2 dz'=h\overline{v_2}+O(\varepsilon). \tag{38}$$

Combining and ordering the terms,

$$\begin{aligned}
&\frac{\partial\eta_1}{\partial t'}+h\frac{\partial u_1}{\partial x'}+\varepsilon\left(\frac{\partial\eta_2}{\partial t'}+h\left(\frac{\partial\overline{u_2}}{\partial x'}+\frac{\partial v_1}{\partial y'}\right)+\frac{\partial}{\partial x'}\left(\eta_1u_1\right)+u_1\frac{\partial h}{\partial\tilde{x}}\right)\\
&+\varepsilon^2\left[\frac{\partial\eta_3}{\partial t'}+h\left(\frac{\partial\overline{u_3}}{\partial x'}+\frac{\partial\overline{v_2}}{\partial y'}\right)+\frac{\partial\left(\eta_1\overline{u_2}\right)}{\partial x'}+\frac{\partial\left(\eta_2u_1\right)}{\partial x'}\right.\\
&\qquad\left.+\frac{\partial}{\partial y'}\left(\eta_1v_1\right)+\overline{u_2}\frac{\partial h}{\partial\tilde{x}}+v_1\frac{\partial h}{\partial\tilde{y}}\right]=O\left(\varepsilon^3\right).
\end{aligned}\tag{39}$$

The system of equations (30), (31), and (39) constitute the full set of governing equations to $O\left(\varepsilon^3\right)$. This system is now cross-differentiated to eliminate η_3, which gives

$$\begin{aligned}
&\frac{\partial^2u_1}{\partial t^2}-\frac{\partial}{\partial x}\left(h\frac{\partial u_1}{\partial x}\right)\\
&+\varepsilon\left[\frac{\partial^2\overline{u_2}}{\partial t^2}-\frac{\partial}{\partial x}\left(h\left(\frac{\partial\overline{u_2}}{\partial x}+\frac{\partial v_1}{\partial y}\right)\right)-\frac{\partial}{\partial x}\left(\frac{h}{3}\frac{\partial^3\left(hu_1\right)}{\partial x^2\partial t^2}\right)\right.\\
&\qquad\left.-\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial\tilde{x}}u_1\right)+\frac{\partial}{\partial t}\left(u_1\frac{\partial u_1}{\partial x}\right)-\frac{\partial^2\left(\eta_1u_1\right)}{\partial x^2}\right]\\
&+\varepsilon^2\left[\frac{\partial^2\overline{u_3}}{\partial t^2}-\frac{\partial}{\partial x}\left(h\left(\frac{\partial\overline{u_3}}{\partial x}+\frac{\partial\overline{v_2}}{\partial y}\right)\right)-\frac{h^2}{3}\left(\frac{\partial^4\overline{u_2}}{\partial x^2\partial t^2}+\frac{\partial^4v_1}{\partial x\partial y\partial t^2}\right)-\frac{h^2}{45}\frac{\partial^6u_1}{\partial x^4\partial t^2}\right.\\
&\qquad-\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial\tilde{x}}\overline{u_2}+\frac{\partial h}{\partial\tilde{y}}v_1\right)-h\frac{\partial^2}{\partial x\partial t}\left(\eta_1\frac{\partial^2u_1}{\partial x\partial t}\right)+\frac{h}{3}\frac{\partial^2}{\partial t^2}\left(\eta_1\frac{\partial^2u_1}{\partial x^2}\right)\\
&\qquad+\frac{\partial}{\partial t}\left(u_1\frac{\partial\overline{u_2}}{\partial x}+\overline{u_2}\frac{\partial u_1}{\partial x}\right)+\frac{\partial}{\partial t}\left(v_1\frac{\partial u_1}{\partial y}\right)+\frac{h^2}{2}\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial u_1}{\partial x}\right)^2\\
&\qquad\left.-\frac{h^2}{3}\frac{\partial}{\partial x\partial t}\left(u_1\frac{\partial^2u_1}{\partial x^2}\right)-\frac{\partial^2}{\partial x^2}\left(\eta_1\overline{u_2}+\eta_2u_1\right)-\frac{\partial}{\partial xdy}\left(\eta_1v_1\right)\right]=O\left(\varepsilon^3\right)
\end{aligned}\tag{40}$$

and

$$
\begin{aligned}
&\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial y}\left(h\frac{\partial u_1}{\partial x}\right) \\
&+\varepsilon\left[\frac{\partial^2 \overline{v_2}}{\partial t^2} - \frac{\partial}{\partial y}\left(h\left(\frac{\partial \overline{u_2}}{\partial x} + \frac{\partial v_1}{\partial y}\right)\right) - \frac{\partial}{\partial y}\left(\frac{h}{3}\frac{\partial^3\left(hu_1\right)}{\partial x \partial t^2}\right)\right. \\
&\qquad \left. - \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial \tilde{x}}u_1\right) + \frac{\partial}{\partial t}\left(u_1\frac{\partial v_1}{\partial x}\right) - \frac{\partial^2\left(\eta_1 u_1\right)}{\partial x \partial y}\right] \\
&+\varepsilon^2\left[\frac{\partial^2 \overline{v_3}}{\partial t^2} - \frac{\partial}{\partial y}\left(h\left(\frac{\partial \overline{u_3}}{\partial x} + \frac{\partial \overline{v_2}}{\partial y}\right)\right) - \frac{h^2}{3}\left(\frac{\partial^4 \overline{u_2}}{\partial x \partial y \partial t^2} + \frac{\partial^4 v_1}{\partial y^2 \partial t^2}\right) - \frac{h^2}{45}\frac{\partial^6 u_1}{\partial x^3 \partial y \partial t^2}\right. \\
&\qquad - \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial \tilde{x}}\overline{u_2} + \frac{\partial h}{\partial \tilde{y}}v_1\right) - h\frac{\partial^2}{\partial y \partial t}\left(\eta_1\frac{\partial^2 u_1}{\partial x \partial t}\right) + \frac{h}{3}\frac{\partial^2}{\partial t^2}\left(\eta_1\frac{\partial^2 u_1}{\partial x \partial y}\right) \\
&\qquad + \frac{\partial}{\partial t}\left(u_1\frac{\partial \overline{v_2}}{\partial x} + \overline{u_2}\frac{\partial v_1}{\partial x}\right) + \frac{\partial}{\partial t}\left(v_1\frac{\partial v_1}{\partial y}\right) + \frac{h^2}{2}\frac{\partial^2}{\partial y \partial t}\left(\frac{\partial u_1}{\partial x}\right)^2 \\
&\qquad \left. - \frac{h^2}{3}\frac{\partial}{\partial y \partial t}\left(u_1\frac{\partial^2 u_1}{\partial x^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(\eta_1\overline{u_2} + \eta_2 u_1\right) - \frac{\partial}{dy^2}\left(\eta_1 v_1\right)\right] = O\left(\varepsilon^3\right), \qquad (41)
\end{aligned}
$$

where the primes are dropped from here on to simplify the notation.

D. GOVERNING EQUATIONS IN VELOCITY POTENTIAL FORM

Finally, expressing the horizontal flow variables as gradients of velocity potential functions ϕ_i (see Appendix B) allows equations (40) and (41) to be converted to the gradient form

$$
\begin{aligned}
\nabla \Bigg\{ & \frac{\partial^2 \phi_1}{\partial t^2} - h\frac{\partial^2 \phi_1}{\partial x^2} \\
& + \varepsilon \Bigg[\frac{\partial^2 \overline{\phi_2}}{\partial t^2} - h\left(\frac{\partial^2 \overline{\phi_2}}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} \right) - \frac{h^2}{3}\frac{\partial^4 \phi_1}{\partial x^2 \partial t^2} \\
& \qquad - \frac{\partial h}{\partial \tilde{x}}\frac{\partial \phi_1}{\partial x} + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial x} \right)^2 - \frac{\partial}{\partial x}\left(\eta_1 \frac{\partial \phi_1}{\partial x} \right) \Bigg] \\
& + \varepsilon^2 \Bigg[\frac{\partial^2 \overline{\phi_3}}{\partial t^2} - h\left(\frac{\partial^2 \overline{\phi_3}}{\partial x^2} + \frac{\partial^2 \overline{\phi_2}}{\partial y^2} \right) - \frac{h^2}{3}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \overline{\phi_2}}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} \right) - \frac{h^4}{45}\frac{\partial^6 \phi_1}{\partial x^4 \partial t^2} \\
& \qquad - \frac{\partial h}{\partial \tilde{x}}\frac{\partial \overline{\phi_2}}{\partial x} - \frac{\partial h}{\partial \tilde{y}}\frac{\partial \phi_1}{\partial y} - h\frac{\partial h}{\partial \tilde{x}}\frac{\partial^3 \phi_1}{\partial x \partial t^2} - h\frac{\partial}{\partial t}\left(\eta_1 \frac{\partial^3 \phi_1}{\partial x^2 \partial t} \right) \\
& \qquad + \frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial x}\frac{\partial \overline{\phi_2}}{\partial x} \right) + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial y} \right)^2 + \frac{h^2}{2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \phi_1}{\partial x^2} \right)^2 - \frac{h^2}{3}\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial x}\frac{\partial^3 \phi_1}{\partial x^3} \right) \\
& \qquad - \frac{\partial}{\partial x}\left(\eta_1 \frac{\partial \overline{\phi_2}}{\partial x} + \eta_2 \frac{\partial \phi_1}{\partial x} \right) - \frac{\partial}{dy}\left(\eta_1 \frac{\partial \phi_1}{\partial y} \right) \Bigg] + O\left(\varepsilon^3 \right) \Bigg\} = 0.
\end{aligned}
\tag{42}
$$

Since equation (42) must hold in the entire horizontal plane, it follows that the expression inside the gradient operator (discarding a uniform time dependence) must vanish. Therefore, the final form of the governing equation is

$$
\begin{aligned}
&\frac{\partial^2 \phi_1}{\partial t^2} - h\frac{\partial^2 \phi_1}{\partial x^2} \\
&+\varepsilon\left[\frac{\partial^2 \overline{\phi_2}}{\partial t^2} - h\left(\frac{\partial^2 \overline{\phi_2}}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}\right) - \frac{h^2}{3}\frac{\partial^4 \phi_1}{\partial x^2 \partial t^2}\right. \\
&\qquad \left. - \frac{\partial h}{\partial \tilde{x}}\frac{\partial \phi_1}{\partial x} + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial x}\right)^2 - \frac{\partial}{\partial x}\left(\eta_1 \frac{\partial \phi_1}{\partial x}\right)\right] \\
&+\varepsilon^2\left[\frac{\partial^2 \overline{\phi_3}}{\partial t^2} - h\left(\frac{\partial^2 \overline{\phi_3}}{\partial x^2} + \frac{\partial^2 \overline{\phi_2}}{\partial y^2}\right) - \frac{h^2}{3}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \overline{\phi_2}}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}\right) - \frac{h^4}{45}\frac{\partial^6 \phi_1}{\partial x^4 \partial t^2}\right. \\
&\qquad - \frac{\partial h}{\partial \tilde{x}}\frac{\partial \overline{\phi_2}}{\partial x} - \frac{\partial h}{\partial \tilde{y}}\frac{\partial \phi_1}{\partial y} - h\frac{\partial h}{\partial \tilde{x}}\frac{\partial^3 \phi_1}{\partial x \partial t^2} - h\frac{\partial}{\partial t}\left(\eta_1 \frac{\partial^3 \phi_1}{\partial x^2 \partial t}\right) \\
&\qquad + \frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial x}\frac{\partial \overline{\phi_2}}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial y}\right)^2 + \frac{h^2}{2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \phi_1}{\partial x^2}\right)^2 - \frac{h^2}{3}\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial x}\frac{\partial^3 \phi_1}{\partial x^3}\right) \\
&\qquad \left. - \frac{\partial}{\partial x}\left(\eta_1 \frac{\partial \overline{\phi_2}}{\partial x} + \eta_2 \frac{\partial \phi_1}{\partial x}\right) - \frac{\partial}{dy}\left(\eta_1 \frac{\partial \phi_1}{\partial y}\right)\right] + O\left(\varepsilon^3\right) = 0.
\end{aligned}
\tag{43}
$$

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III. THIRD ORDER AMPLITUDE EVOLUTION EQUATION

A. ASSUMED SPECTRAL WAVEFORM

In the previous chapter, we derived a governing equation (43) for the depth-averaged velocity potential function to $O\left(\varepsilon^3\right)$. We now derive spectral amplitude evolution equations by assuming the lowest order wave field $\left(\eta_1, \phi_1\right)$ to be a linear superposition of nearly plane, shoreward-propagating waves

$$\eta_1 = \sum_p \sum_q A_{p,q}\left(\tilde{x}, \tilde{y}\right) \exp\left[i\varphi_{p,q}\left(x, y, t\right)\right], \tag{44}$$

$$\phi_1 = \sum_p \sum_q \frac{A_{p,q}\left(\tilde{x}, \tilde{y}\right)}{i\omega_p} \exp\left[i\varphi_{p,q}\left(x, y, t\right)\right], \tag{45}$$

where $\omega_p = p\Delta\omega$ and $l_q = q\Delta l$ are the scaled frequency and initial alongshore wavenumber, with $\Delta\omega$ and Δl the separation of adjacent bands in the Fourier representation. The phase function $\varphi_{p,q}$ contains the fast variations of the lowest-order wave field,

$$\varphi_{p,q}\left(x, y, t\right) = \int_0^x \frac{\omega_p}{h^{1/2}\left(\xi, y\right)} d\xi + l_q y - \omega_p t. \tag{46}$$

The complex amplitude $A_{p,q}$ is a slow function of x and y

$$\frac{\partial A_{p,q}}{\partial x} = \varepsilon R_{1p,q} + \varepsilon^2 R_{2p,q} + O\left(\varepsilon^3\right), \qquad \frac{\partial A_{p,q}}{\partial y} = \varepsilon S_{1p,q} + O\left(\varepsilon^2\right), \tag{47}$$

where $R_{1p,q}$, $R_{2p,q}$, and $S_{1p,q}$ account for amplitude and phase changes resulting from linear (e.g., dispersion, shoaling, refraction) and nonlinear (triad and quartet interactions) processes, and changes in l_q associated with alongshore depth variations are absorbed in the slow alongshore variations of $A_{p,q}$. It will be shown that $\partial A_{p,q}/\partial y$ terms appear at sufficiently high order that they do not require explicit expression beyond $O(\varepsilon)$.

B. $O(1)$ PROBLEM

Define the linear operator $\mathcal{L}$ as

$$\mathcal{L} \equiv \frac{\partial^2}{\partial t^2} - h\frac{\partial^2}{\partial x^2}. \tag{48}$$

It is easily verified that ϕ_1 is a solution to

$$\mathcal{L}(\phi_1) = O(\varepsilon). \tag{49}$$

To determine solutions to higher order problems, the residual right hand side of (49) must be evaluated. To $O(\varepsilon^3)$, $\mathcal{L}(\phi_1)$ is given by equation (91) in Appendix C.

C. $O(\varepsilon)$ PROBLEM

At $O(\varepsilon)$, applying the results of Appendix C,

$$\begin{aligned}\mathcal{L}\left(\overline{\phi_2}\right) = \sum_p \sum_q \Bigg[& 2R_{1p,q}h^{1/2} + A_{p,q}\frac{\partial h}{\partial \tilde{x}} + i\left(\frac{-h\omega_p^3 A_{p,q}}{3} + \frac{hl_q^2 A_{p,q}}{\omega_p}\right) \\ & + \frac{3\omega_p}{2h}\sum_m \sum_n A_{m,n}A_{p-m,q-n}\Bigg]\exp\left[i\varphi_{p,q}\right] + O(\varepsilon).\end{aligned} \tag{50}$$

Resonant growth of $\overline{\phi_2}$ is prevented by the solubility constraint that the right-hand side forcing terms in (50) do not contain any free wave solutions of the general form $F\left(x/h^{1/2} \pm t\right)$. Since all these terms obey to $O(\varepsilon)$ the shallow water dispersion relation, it follows that the right hand side of (50) must vanish. Solving for $R_{1p,q}$ yields the lowest order term in the order amplitude evolution equation:

$$R_{1p,q} = L_{1p,q} A_{p,q} + Q_{1p} \sum_r \sum_s A_{r,s} A_{p-r,q-s} \tag{51}$$

where

$$L_{1p,q} = \frac{-1}{4h}\frac{\partial h}{\partial \tilde{x}} + i\left(\frac{h^{1/2}\omega_p^3}{6} - \frac{h^{1/2} l_q^2}{2\omega_p}\right),$$
$$Q_{1p=} - i\frac{3\omega_p}{4h^{3/2}}.$$

This result was derived in HB.

D. $O\left(\varepsilon^2\right)$ PROBLEM

To determine the remaining higher order amplitude evolution terms $R_{2p,q}$ and $S_{1p,q}$, we proceed in the same fashion as for $R_{1p,q}$. Since $\mathcal{L}\left(\overline{\phi_2}\right) = O(\varepsilon)$, $\overline{\phi_2}$ represents (to lowest order) free shallow water wave motion that can be absorbed in ϕ_1., and thus $\overline{\phi_2}$ can be set equal to zero. Note, however, that η_2 (evaluated in Appendix D) contains deviations from free shallow water wave motion that contribute to the forcing of $\overline{\phi_3}$. Tedious but straightforward evaluation of all the $O\left(\varepsilon^2\right)$ forcing terms in equation (43) (see Appendix C) yields

$$
\begin{aligned}
&\mathcal{L}\left(\overline{\phi_3}\right)= \\
&\quad\sum_p\sum_q\Bigg\{-2h^{1/2}R_{2p,q}-\frac{2hl_q}{\omega_p}S_{1p,q}+\frac{ih}{\omega_p}\frac{\partial R_{1p,q}}{\partial\tilde{x}}+\left(\frac{2h^{3/2}\omega_p^2}{3}+\frac{i}{\omega_p}\frac{\partial h}{\partial\tilde{x}}\right)R_{1p,q} \\
&\quad+\left[\left(\frac{5h^{1/2}\omega_p^2}{6}\frac{\partial h}{\partial\tilde{x}}-\frac{l_q}{\omega_p}\frac{\partial h}{\partial\tilde{y}}\right)+i\left(\frac{-h^2\omega_p^5}{45}+\frac{h^2\omega_p l_q^2}{3}\right)\right]A_{p,q} \qquad (52) \\
&\quad+\sum_r\sum_s\left[-\frac{\omega_p}{\omega_r}\left(\frac{A_{r,s}}{2h^{3/2}}\frac{\partial h}{\partial\tilde{x}}+\frac{2R_{1r,s}}{h^{1/2}}\right)+i\left(\frac{5\omega_p\omega_r^2}{3}+\frac{\omega_{p-r}l_s^2-\omega_r l_q l_s}{\omega_r^2}\right)A_{r,s}\right]A_{p-r,q-s} \\
&\quad+\sum_r\sum_s\sum_m\sum_n\left[\frac{-i\omega_p}{h^2}\right]A_{m,n}A_{r,s}A_{p-r-m,q-s-n}\Bigg\}\exp\left[i\varphi_{p,q}\right]+O(\varepsilon).
\end{aligned}
$$

Finally, substituting $R_{1p,q}$ (equation (51)) and

$$
\begin{aligned}
\frac{\partial R_{1p,q}}{\partial\tilde{x}}&=\left[\frac{-1}{4h}\frac{\partial^2 h}{\partial\tilde{x}^2}+\frac{5}{16}\left(\frac{\partial h}{\partial\tilde{x}}\right)^2+h\left(-\frac{\omega_p^6}{36}+\frac{\omega_p^2 l_q^2}{6}-\frac{l_q^4}{4\omega_p^2}\right)\right]A_{p,q} \\
&+\sum_r\sum_s\left[\frac{3\omega_p}{4h}\left(\frac{1}{2}\left(\frac{\omega_p^3}{3}-\frac{l_q^2}{\omega_p}\right)+\left(\frac{\omega_r^3}{3}-\frac{l_s^2}{\omega_r}\right)\right)+i\frac{27\omega_p}{16h^{5/2}}\frac{\partial h}{\partial\tilde{x}}\right]A_{r,s}A_{p-r,q-s} \\
&+\sum_r\sum_s\sum_m\sum_n\left[-\frac{9\omega_p\omega_{p-r}}{8h^3}\right]A_{m,n}A_{r,s}A_{p-r-m,q-s-n}+O(\varepsilon)
\end{aligned}
$$

in equation (52) and setting the forcing terms on the right hand side equal to zero to remove all secular terms, the following relation for the higher order amplitude gradients $R_{2p,q}$ and $S_{1p,q}$ is obtained:

$$
\begin{aligned}
R_{2p,q}+\frac{h^{1/2}l_q}{\omega_p}S_{1p,q}&=L_{2p,q}A_{p,q}+\sum_r\sum_s Q_{2p,q,r,s}A_{r,s}A_{p-r,q-s} \\
&\quad+\sum_r C_{2p,r}\sum_s\sum_m\sum_n A_{m,n}A_{r,s}A_{p-r-m,q-s-n},
\end{aligned}
\qquad (53)
$$

where

$$
\begin{aligned}
L_{2p,q} &= \frac{1}{4}\left(\omega_p^2 + \frac{l_q^2}{\omega_p^2}\right)\frac{\partial h}{\partial \tilde{x}} - \frac{l_q}{2h^{1/2}\omega_p}\frac{\partial h}{\partial \tilde{y}} \\
&\quad + i\left[h^{3/2}\left(\frac{11\omega_p^5}{360} + \frac{\omega_p l_q^2}{12} - \frac{l_q^4}{8\omega_p^3}\right) + \frac{1}{32h^{3/2}\omega_p}\left(\frac{\partial h}{\partial \tilde{x}}\right)^2 - \frac{1}{8h^{1/2}\omega_p}\frac{\partial^2 h}{\partial \tilde{x}^2}\right], \\
Q_{2p,q,r,s} &= \frac{-15}{32h^2}\frac{\partial h}{\partial \tilde{x}} + i\left[\frac{3}{16h^{1/2}}\left(\frac{\omega_p^3}{3} - \frac{l_q^2}{\omega_p}\right) + \frac{3}{8h^{1/2}}\left(\frac{\omega_r^3}{3} - \frac{l_s^2}{\omega_r}\right)\right. \\
&\qquad\qquad \left. + \frac{-3\omega_p^3 + 8\omega_p\omega_r^2}{12h^{1/2}} + \frac{\left(3\omega_p - \omega_r\right)l_s^2 - \omega_r l_q l_s}{2h^{1/2}\omega_r^2}\right], \\
C_{2p,r} &= i\frac{-5\omega_p + 9\omega_r}{16h^{5/2}}.
\end{aligned}
$$

E. EVOLUTION EQUATION

Combining (47), (51), and (53), our final result in amplitude form is

$$
\begin{aligned}
\frac{\partial A_{p,q}}{\partial x} + \frac{h^{1/2}l_q}{\omega_p}\frac{\partial A_{p,q}}{\partial y} &= \left(L_{1p,q} + L_{2p,q}\right)A_{p,q} + \sum_r\sum_s\left(Q_{1p} + Q_{2p,q,r,s}\right)A_{r,s}A_{p-r,q-s} \\
&\quad + \sum_r C_{2p,r}\sum_s\sum_m\sum_n A_{m,n}A_{r,s}A_{p-r-m,q-s-n} + O\left(\varepsilon^3\right).
\end{aligned}
\tag{54}
$$

The real part of the linear term on the right hand side is the amplitude growth of the given mode due to shoaling (i.e., changes in the group speed); the imaginary part represents slow phase changes owing to dispersion, directional spreading, and bottom slope effects. The double and quadruple sums contain the slow amplitude and phase changes of a mode resulting from nonlinear interactions of all possible triads and quartets (respectively) in which the mode participates.

The new terms (relative to HB: $L_{2p,q}$, $Q_{2p,q,r,s}$, and $C_{2p,r}$) introduce improved representations of dispersion, directional spreading, triad interactions, and the effects of the cross-shore bottom slope (all to one higher order of accuracy), as well as the lowest order effects of alongshore depth variations and nonlinear quartet interactions not represented in HB. The verification of many of these new terms, and a quantitative investigation of the improvements gained by them, is given in the next chapter.

IV. VERIFICATION OF LINEARIZED MODEL FOR AN ALONGSHORE UNIFORM BEACH

A. VELOCITY PROFILES

From linear finite depth theory, the horizontal velocity profile (in the *x*-direction) can be expressed in normalized form as

$$\frac{u}{\overline{u}}=\frac{\cosh\left(\kappa\left(z+h\right)\right)}{\frac{1}{h}\int_{-h}^{0}\cosh\left(\kappa\left(z+h\right)\right)dz}=\frac{\kappa h\cosh\left(\kappa\left(z+h\right)\right)}{\sinh\left(\kappa h\right)}. \tag{55}$$

Expanding the hyperbolic functions for small κh and using the geometric expansion for the quotient in equation (55) yields

$$\begin{aligned}\frac{u}{\overline{u}}&=\frac{\kappa h\left(1+\frac{\kappa^2\left(z+h\right)^2}{2}+\frac{\kappa^4\left(z+h\right)^4}{24}+O\left(\kappa h\right)^6\right)}{\kappa h\left(1+\frac{\left(\kappa h\right)^2}{6}+\frac{\left(\kappa h\right)^4}{120}+O\left(\kappa h\right)^6\right)}\\&=\left(1+\kappa^2\left(\frac{z^2}{2}+hz+\frac{h^2}{2}\right)+\kappa^4\left(\frac{z^4}{24}+\frac{hz^3}{6}+\frac{h^2z^2}{4}+\frac{h^3z}{6}+\frac{h^4}{24}\right)\right)\\&\quad\times\left(1-\frac{\left(\kappa h\right)^2}{6}-\frac{\left(\kappa h\right)^4}{120}+\frac{\left(\kappa h\right)^4}{36}\right)+O\left(\kappa h\right)^6.\end{aligned}$$

Collecting and combining like terms,

$$\frac{u}{\overline{u}}=1+\kappa^2\left(\frac{z^2}{2}+hz+\frac{h^2}{3}\right)+\kappa^4\left(\frac{z^4}{24}+\frac{hz^3}{6}+\frac{h^2z^2}{6}-\frac{h^4}{45}\right)+O\left(\kappa h\right)^6. \tag{56}$$

The z-polynomials at each order in equation (56) are in exact agreement with the polynomial coefficients at the corresponding orders for u_2, v_2, u_3, and v_3 (equations (18), (19), (23), and (25)). Figure 4.1 compares vertical structure of the first $\left(u_1\right)$, second

$(u_1 + \varepsilon u_2)$, and third $(u_1 + \varepsilon u_2 + \varepsilon^3 u_3)$ order approximations of u with the exact linear finite depth solution (55) for a wave with period 10s in 20m water depth. The first order approximation is the familiar depth-independent shallow water solution for the horizontal flow. The second order approximation better represents the depth dependence of the horizontal flow, while the third order approximation virtually replicates the exact finite depth solution.

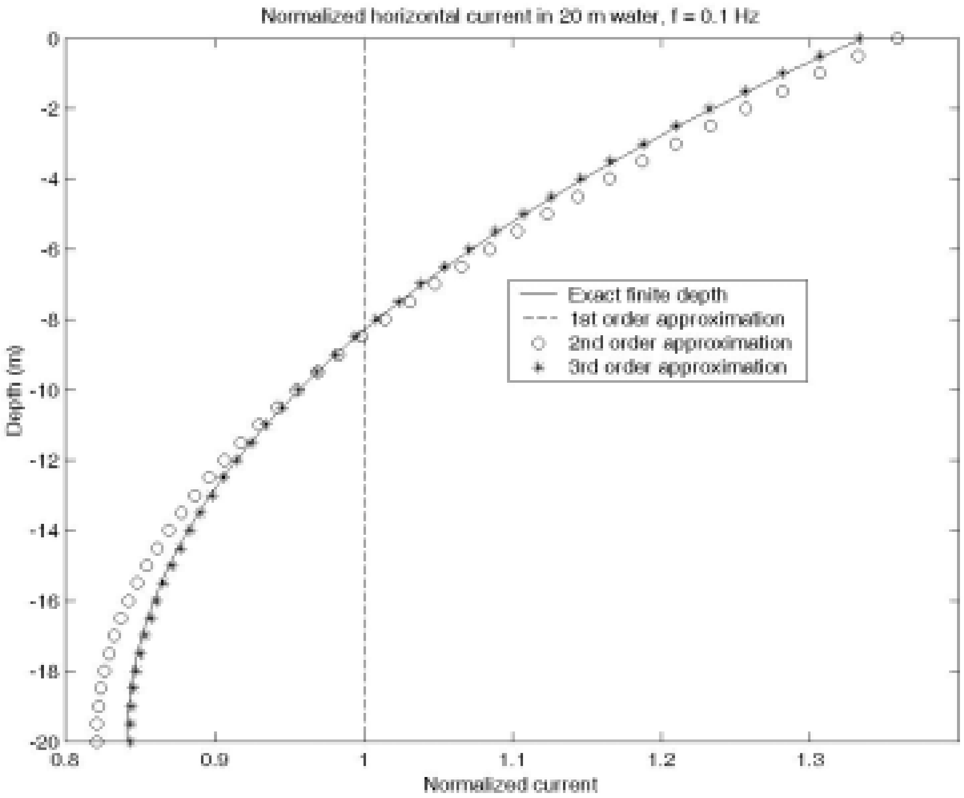

Figure 4.1. Vertical profiles of normalized horizontal current . Note that the first order estimate is depth independent. Higher order estimates converge toward the vertical structure predicted by finite depth theory.

B. DISPERSION

In linearized form , the amplitude evolution equation for an alongshore uniform beach $(\partial h/dy = 0,\ \partial A/dy = 0)$ is given by

$$\begin{aligned}\frac{\partial A_{p,q}}{\partial x} = \Bigg\{&\left(\frac{-1}{4h}+\frac{1}{4}\left(\omega_p^2+\frac{l_q^2}{\omega_p^2}\right)\right)\frac{\partial h}{\partial \tilde{x}} \\ &+i\left[\frac{h^{1/2}\omega_p^3}{6}-\frac{h^{1/2}l_q^2}{2\omega_p}+h^{3/2}\left(\frac{11\omega_p^5}{360}+\frac{\omega_p l_q^2}{12}-\frac{l_q^4}{8\omega_p^3}\right)+\frac{1}{32h^{3/2}\omega_p}\left(\frac{\partial h}{\partial \tilde{x}}\right)^2-\frac{1}{8h^{1/2}\omega_p}\frac{\partial^2 h}{\partial \tilde{x}^2}\right]\Bigg\} A_{p,q}.\end{aligned} \tag{57}$$

The imaginary part of the right hand side represents slow phase changes as the wave component evolves in the cross-shore direction, that is, corrections to the cross-shore wavenumber, *k*. From equation (57) it follows that *k* for a wave with radial frequency ω and alongshore wavenumber *l* in uniform depth is given to $O(\kappa h)^4$ by

$$k = \frac{\omega}{h^{1/2}}+\frac{h^{1/2}\omega^3}{6}+\frac{11h^{3/2}\omega^5}{360}-\frac{h^{1/2}l^2}{2\omega}+\frac{h^{3/2}\omega l^2}{12}-\frac{h^{3/2}l^4}{8\omega^3}. \tag{58}$$

Now consider the exact *k* in finite depth. The alongshore and cross-shore wavenumbers satisfy the relationship

$$k^2+l^2=\kappa^2. \tag{59}$$

Using the finite depth dispersion relation, the cross-shore wavenumber can be expressed as

$$k=\sqrt{\frac{\omega^4}{\tanh^2(\kappa h)}-l^2}. \tag{60}$$

Since *l* is $O(\kappa h)^2$, equation (60) can be written as the following Taylor series expansion:

$$k=\frac{\omega^2}{\tanh(\kappa h)}-\frac{l^2\tanh(\kappa h)}{2\omega^2}-\frac{l^4\tanh^3(\kappa h)}{8\omega^6}+O(\kappa h)^6. \tag{61}$$

Now, expanding the hyperbolic functions on the right hand side of equation (61) in terms of κh to $O(\kappa h)^4$, and then expressing each κh term as a function of ω and h using the expansion of the finite depth dispersion relation,

$$\omega^2 = \kappa^2 h \left(1 - \frac{(\kappa h)^2}{3} + \frac{2(\kappa h)^4}{15} \right) + O(\kappa h)^6 , \tag{62}$$

we obtain

$$k = \frac{\omega}{h^{1/2}} + \frac{h^{1/2}\omega^3}{6} + \frac{11h^{3/2}\omega^5}{360} - \frac{h^{1/2}l^2}{2\omega} + \frac{h^{3/2}\omega l^2}{12} - \frac{h^{3/2}l^4}{8\omega^3} \tag{63}$$

in agreement with equation (58). Figure 4.2 shows the first ($O(1)$), second ($O(\kappa h)^2$), and third ($O(\kappa h)^4$) order approximations of the cross-shore wavenumber compared to the exact finite depth solution for a normally incident wave with a period of 10s. Figure 4.3 shows the same comparisons for an incidence angle of 30 degrees.

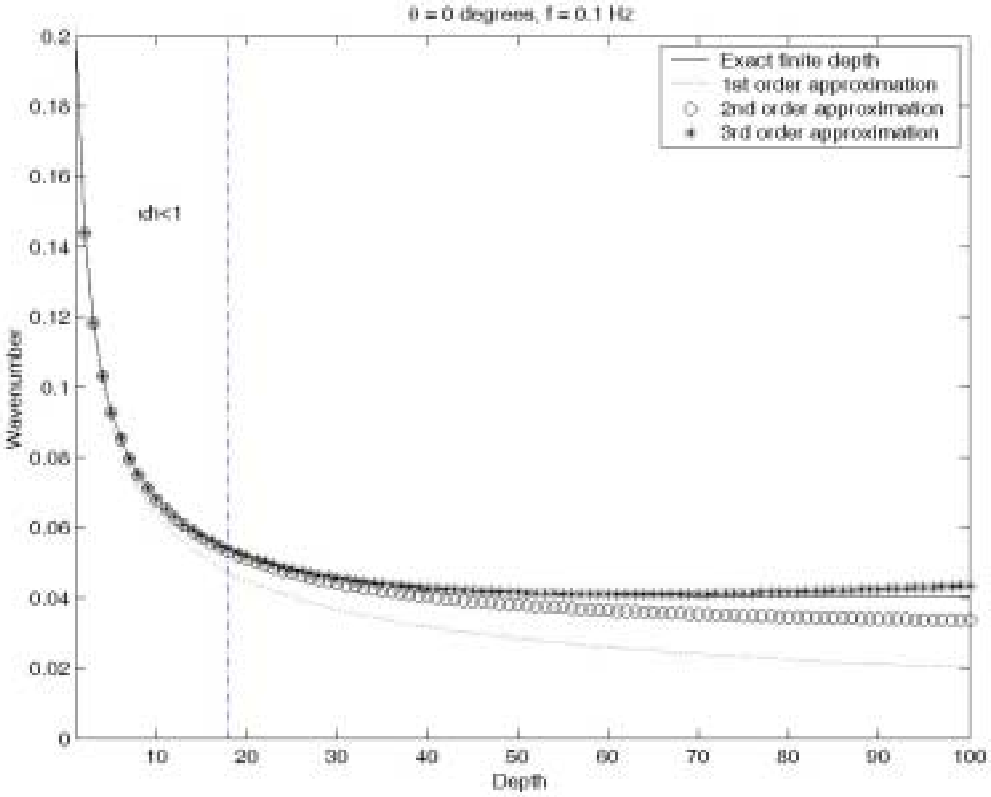

Figure 4.2. Cross-shore wavenumber vs. depth for the first, second, and third order approximations are compared to the exact solution. Results are shown for a 10s period wave at normal incidence.

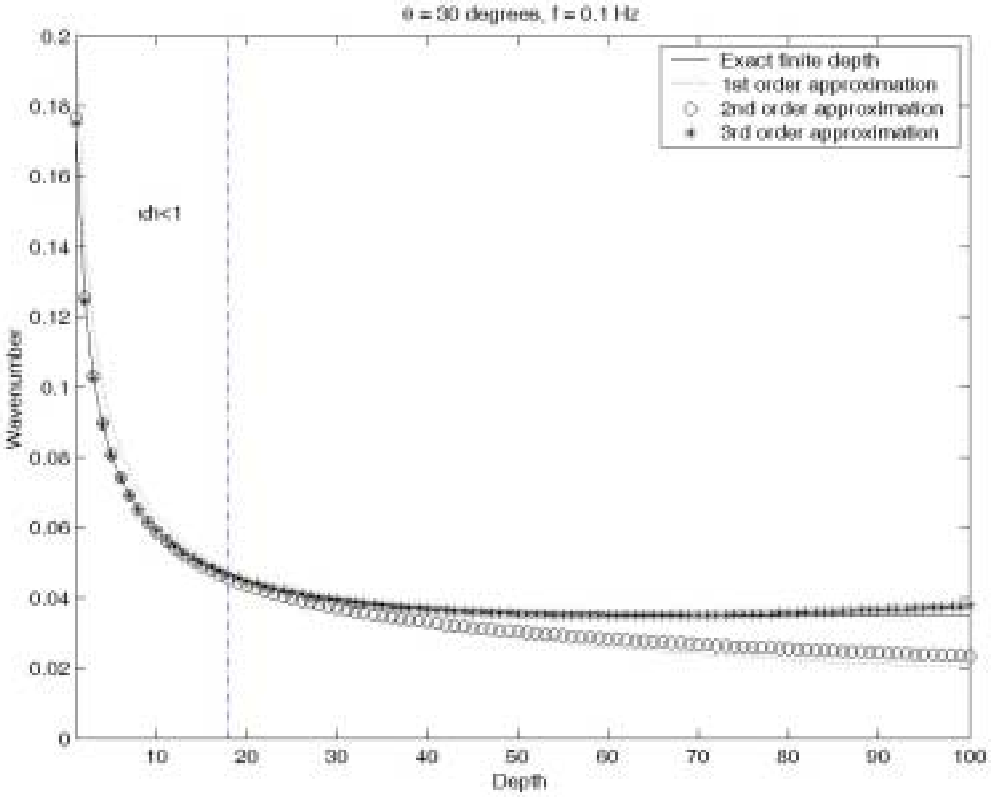

Figure 4.3. Cross-shore wavenumber vs. depth for the first, second, and third order approximations are compared to the exact solution. Results are shown for a 10s period wave at a 30 degree angle.

Though not a significant improvement for depths less than 20m where $\kappa h < 1$, the third order approximation does show a notable improvement over the second order for a wide range of intermediate (40-100m) depths. Note that the third order approximation eventually diverges from the exact result in sufficiently deep depths; as expected from a Taylor series expansion for small κh, this divergence grows rapidly with depth compared to the second order result.

Figure 4.4 shows the same approximations of cross-shore wavenumber k compared to the exact finite depth solution for a 10s wave in a fixed depth of 20m, versus incidence angles.

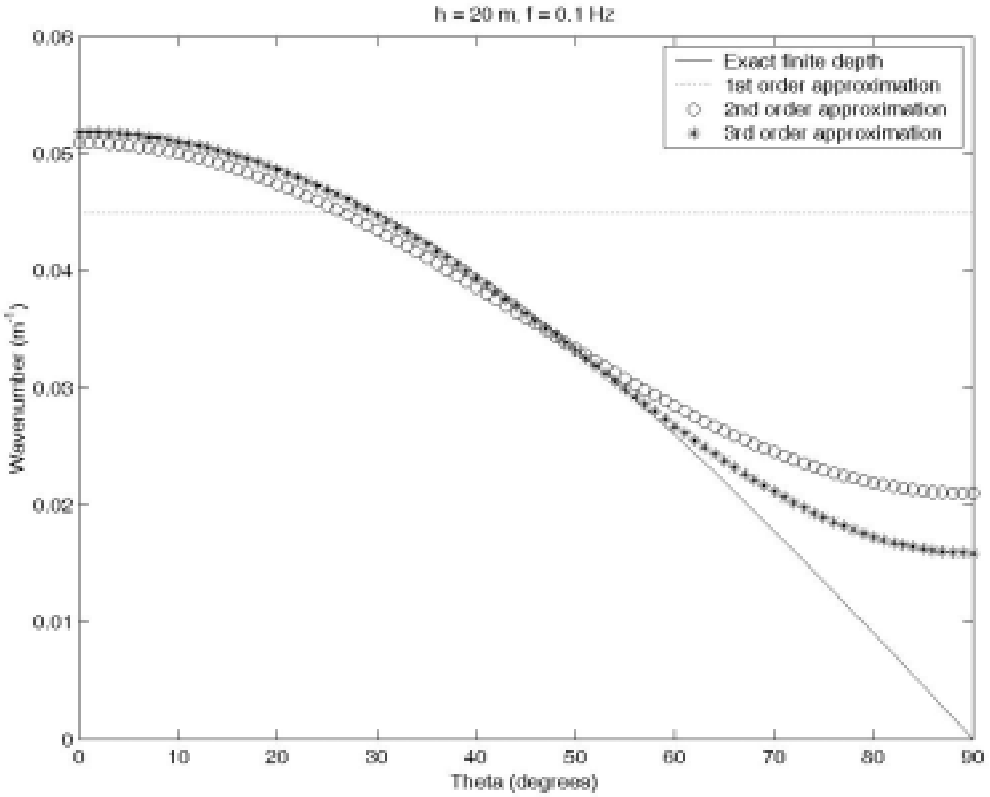

Figure 4.4. Cross-shore wavenumber *k* vs. incidence angle θ_0. The first, second, and third order approximations are compared to the exact linear solution. Results are shown for a 10s period wave in 20m depth.

The first order approximation does not account for directional effects and has large errors. Both the second and third order approximations accurately account for directionality. The third order approximation is within a few percent of the exact solution even for angles up to 60 degrees.

C. SHOALING

The conservation of energy for a wave train with radial frequency ω and alongshore wavenumber *l* may be expressed as

$$\frac{\partial}{\partial x}\left(Ec_{gx}\right) = 0,$$

or

$$\frac{\partial E}{\partial x} = \left(-\frac{1}{c_g}\frac{\partial c_g}{\partial x} - \left(\frac{l^2}{\kappa^2 - l^2} \right) \frac{1}{\kappa}\frac{\partial \kappa}{\partial x} \right) E. \tag{64}$$

where $E = |A|^2$ is proportional to the wave energy, and c_{gx} is the cross-shore component of the group speed c_g. For arbitrary finite depth, the (non-dimensional) linear dispersion relation is given by $\omega^2 = \kappa \tanh(\kappa h)$, yielding the following analytical expressions for c_g, $\frac{\partial c_g}{\partial x}$, and $\frac{\partial \kappa}{\partial x}$:

$$c_g \equiv \frac{\partial \omega}{\partial \kappa} = \frac{\omega}{\kappa}\left(\frac{1}{2} + \frac{\kappa h}{\sinh(\kappa h)} \right), \tag{65}$$

$$\frac{\partial c_g}{\partial x} = \left(-\frac{\omega}{2\kappa^2} - \frac{2\omega h^2}{\tanh^2(2\kappa h)\sinh(2\kappa h)} \right) \frac{\partial \kappa}{\partial x} + \frac{\omega}{\sinh(2\kappa h)}\left(1 - \frac{2\kappa h}{\tanh(2\kappa h)} \right) \frac{\partial h}{\partial x}, \tag{66}$$

$$\frac{\partial \kappa}{\partial x} = \frac{-\kappa^2}{\frac{1}{2}\sinh^2(2\kappa h) + \kappa h}\frac{\partial h}{\partial x}. \tag{67}$$

Substituting equations (65)-(67) into equation (64) yields

$$\begin{aligned}\frac{\partial E}{\partial x} = \Bigg(& -\frac{\kappa \sinh(2\kappa h)}{2\left(\frac{1}{2}\sinh(2\kappa h) + \kappa h\right)^2} - \frac{2\kappa(\kappa h)^2}{\tanh(2\kappa h)\left(\frac{1}{2}\sinh(2\kappa h) + \kappa h\right)^2} \\ & - \frac{\kappa\left(\tanh(2\kappa h) - 2\kappa h\right)}{\tanh(2\kappa h)\left(\frac{1}{2}\sinh(2\kappa h) + \kappa h\right)} + \frac{\kappa l^2}{\left(\kappa^2 - l^2\right)\left(\frac{1}{2}\sinh(2\kappa h) + \kappa h\right)} \Bigg) \frac{\partial h}{\partial x} E.\end{aligned} \tag{68}$$

In order to compare the exact, linear energy equation with the Boussinesq form (54), we expand equation (68) in a Taylor series for small κh to $O(\kappa h)^2$, which yields

$$\frac{\partial E}{\partial x}=\left(-\frac{1}{2h}+\frac{(\kappa h)^2}{2h}+\frac{1}{2h}\left(\frac{l^2}{\kappa^2-l^2}\right)\right)\frac{\partial h}{\partial x}E. \tag{69}$$

Now, since $E=|A|^2=AA^*$ (where "*" denotes complex conjugation), the amplitude evolution equation (57) yields the following energy equation (dropping the *p,q* indices)

$$\frac{\partial E}{\partial x}=A\frac{\partial A^*}{\partial x}+A^*\frac{\partial A}{\partial x}=\left[-\frac{1}{2h}+\varepsilon\left(\frac{\omega^2}{2}+\frac{l^2}{\omega^2}\right)\right]\frac{\partial h}{\partial x}E. \tag{70}$$

By observation, it is immediately seen that equation (69) and equation (70) agree to first order. To compare the $O(\varepsilon)$ terms, note that $\omega^2=\kappa^2 h\left(1+O(\varepsilon)\right)$ and $\kappa^2-l^2=\kappa^2\left(1+O(\varepsilon)\right)$, and thus the $O(\varepsilon)$ terms in equation (70) are in agreement with equation (69) to $O(\varepsilon^2)$.

The improved energy conservation properties of the present model can be observed in Figures 4.5 and 4.6, which compare the second and third order approximations of the energy flux coefficient $-\frac{1}{c_{gx}}\frac{\partial c_{gx}}{\partial x}$ to the exact linear finite depth solution. Again, results are shown for a 10s wave at normal incidence, Figure 4.5, and for a 30 degree incidence angle, Figure 4.6.

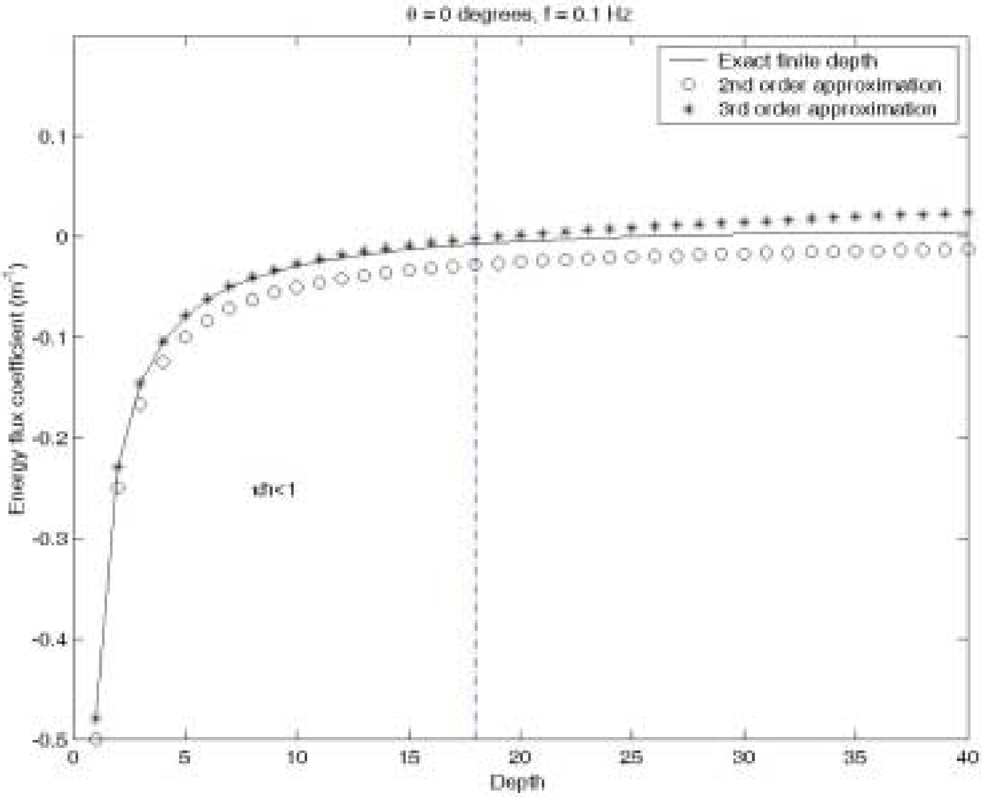

Figure 4.5. Energy flux coefficient $-\frac{1}{c_{gx}}\frac{\partial c_{gx}}{\partial x}$ vs. depth for the second and third order approximations are compared to the exact solution. Results are shown for a 10s period wave at normal incidence.

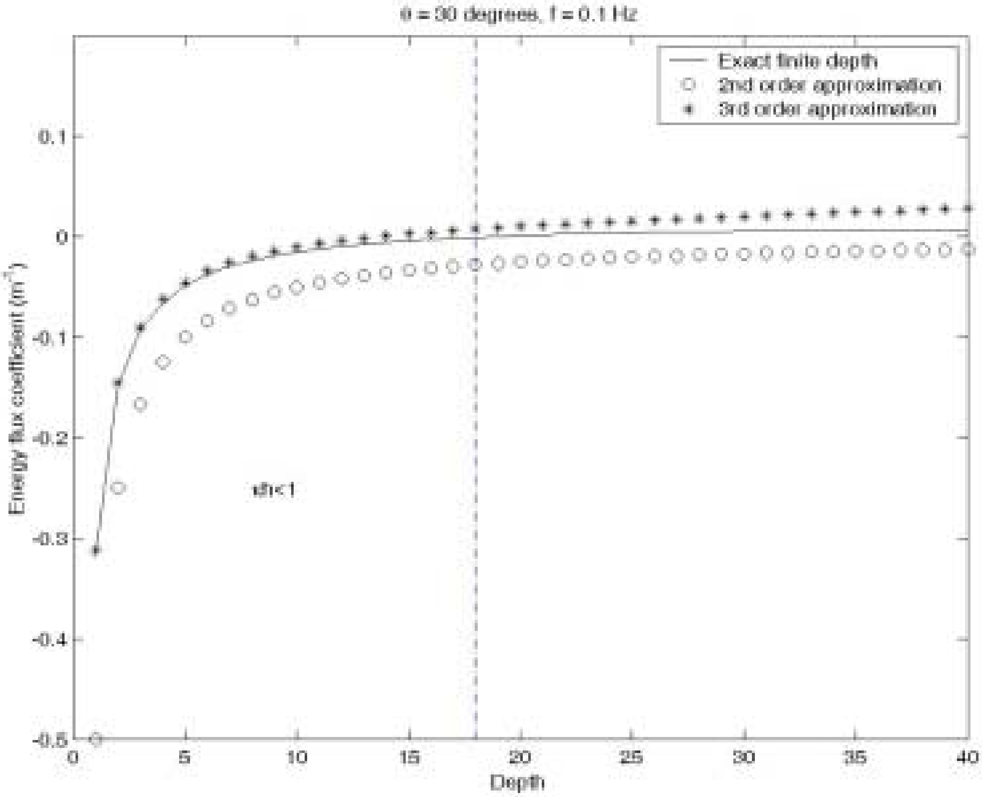

Figure 4.6. Energy flux coefficient $-\frac{1}{c_{gx}}\frac{\partial c_{gx}}{\partial x}$ vs. depth for the second and third order approximations are compared to the exact solution. Results are shown for a 10s period wave at a 30 degree angle.

Note that the second order approximation diverges from the exact solution even in relatively shallow water (<20m), and that this divergence is even more extreme for oblique waves. The third order approximation shows significant improvement over the second order in both the normal and oblique cases.

To illustrate the improved shoaling characteristics of the third order model, the amplitude evolution of a wave shoaling on a plane beach was predicted numerically using a simple first-order finite difference approximation of the amplitude evolution equation. Figure 4.7 compares the evolution of $|A/A_0|$ (where A_0 is the complex amplitude of the incident deep-water wave) predicted by the second (HB) and third order model with the exact finite depth solution.

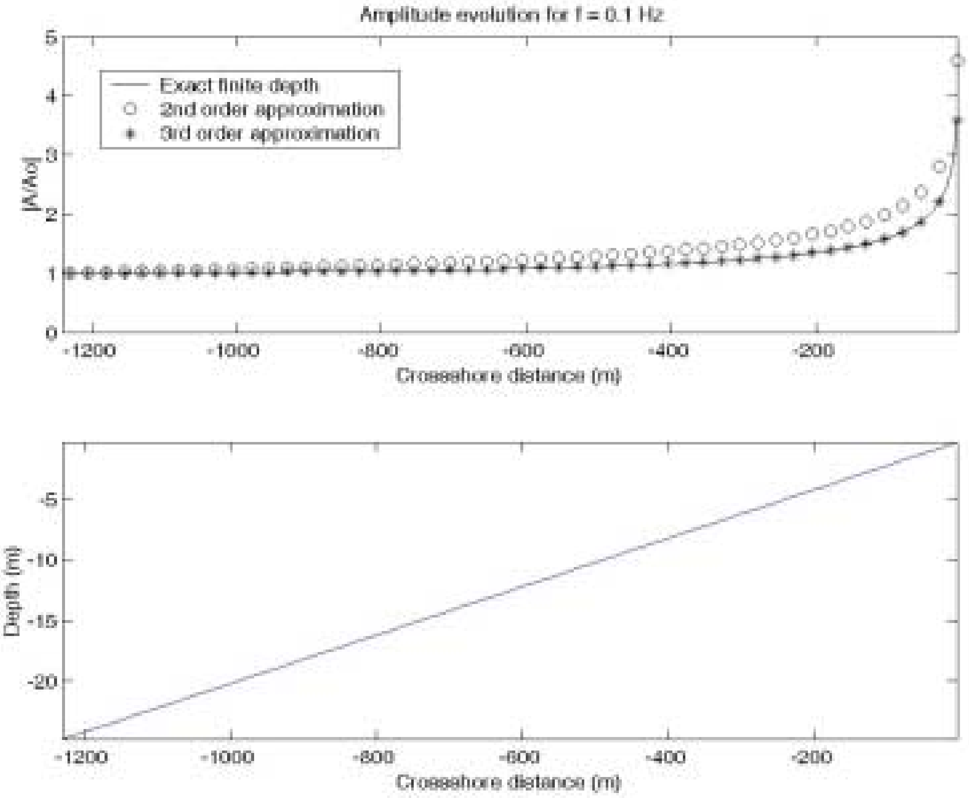

Figure 4.7. Shoaling of a 10s period wave normally incident on a plane beach with a 1:50 slope. Topography varies in the cross-shore direction only. Top panel compares the second and third order model results to finite depth theory. Bottom panel shows the depth profile.

The significant improvement of the third order model over the second order HB model is evident in the predicted amplitudes in shallow water. For this relatively deep (25m) model initialization, the second order model shows the expected overshoaling, overpredicting the wave amplitude in 1m depth by 28%; whereas the third order model is within 1% of the exact finite depth theory prediction.

V. SUMMARY AND FUTURE RESEARCH

In general, wave transformation on beaches is well-described by one-dimensional models. However, the directionality of waves is of crucial importance to a variety of nearshore processes including infragravity motions, alongshore currents, and sediment transport. Additionally, alongshore varying bathymetry complicates nearshore wave evolution. In this thesis, Boussinesq-type equations for weakly nonlinear, weakly dispersive waves are derived for a beach with weak alongshore depth variations.

We extend the derivation of Herbers and Burton (1997) to one higher order in nonlinearity, dispersion, and bottom slope to derive a coupled set of amplitude evolution equations for a spectrum of waves. The main result, equation (54), more accurately represents dispersion, directional spreading, and the strength of triad interactions than the HB model, and extends the effects of cross-shore bottom slope to one higher order. New effects introduced at the lowest order are alongshore depth variations and nonlinear quartet interactions. Linear terms were verified by analytical methods, and example computations in uniform depth and for a plane beach quantify some of the improvements of this model over the lower order HB model. In particular, it is demonstrated that this higher order model better represents the vertical structure of the flow field, notably improves the approximation of the cross-shore wave number both in intermediate depths and for larger directional spreading angles, and more accurately accounts for energy conservation. For a typical swell period and model initialization in relatively deep (continental shelf) depths, it is shown that the overprediction of shoaling is reduced from $O(30\%)$ for the HB model to less than 1% for the present higher order model.

Further work is needed to verify other terms in equation (54). No simple analytical results exist for beaches with two-dimensional depth variations. Therefore, the term that accounts for alongshore depth variation may have to be verified numerically using an appropriate refraction model. Higher order cross-shore bottom slope terms in equation (54) involving the square and the curvature of the bottom slope are perhaps more difficult to verify. However, Chu & Mei (1970) derived similar terms for a Stokes wave in arbitrary depth. Their expressions are rather complicated in terms of hyperbolic functions. An expansion of the Chu and Mei result for small κh to $O\left(\kappa h\right)^4$ should render terms that correspond to the cross-shore bottom slope terms derived in this work.

Finally, the coupling coefficients for triad and quartet interactions remain to be verified. HB show that in the limit of small wave amplitude and bottom slope the steady solutions to their evolution equations exactly match the second order bound waves predicted by finite depth theory in the shallow water limit. It should be possible (although more algebraically tedious) to extend this approach to one higher order and validate both corrections to the triad coupling coefficient and the quartet coupling coefficient in equation (54).

APPENDIX A

The average horizontal flow $\left(\overline{u},\overline{v}\right)$ over the water column is given by

$$\begin{aligned}\overline{u} &\equiv \frac{1}{h+\eta}\int_{-h}^{\eta} u\partial z' = \varepsilon u_1 + \varepsilon^2\overline{u_2} + \varepsilon^3\overline{u_3} + O\left(\varepsilon^4\right),\\ \overline{v} &\equiv \frac{1}{h+\eta}\int_{-h}^{\eta} v\partial z' = \sigma\left(\varepsilon v_1 + \varepsilon^2\overline{v_2} + \varepsilon^3\overline{v_3} + O\left(\varepsilon^4\right)\right).\end{aligned} \tag{71}$$

The reciprocal of the water column height can be expressed by geometric expansion as

$$\frac{1}{h+\eta} = \frac{1}{h} - \varepsilon\frac{\eta_1}{h^2} + \varepsilon^2\left(\frac{\eta_1^2}{h^3} - \frac{\eta_2}{h^2}\right) + O\left(\varepsilon^3\right).$$

The integral term of $\overline{u}$ is expanded (note u_1 is independent of z')

$$\begin{aligned}\int_{-h}^{\eta} u\partial z' &= \int_{-h}^{\eta}\left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + O\left(\varepsilon^4\right)\right)\partial z'\\ &= \left(h + \varepsilon\eta_1 + \varepsilon^2\eta_2 + O\left(\varepsilon^3\right)\right)\varepsilon u_1\\ &\quad + \varepsilon^2\left(\int_{-h}^{0} u_2\partial z' + \int_{0}^{\varepsilon\eta_1} u_2\partial z'\right) + \varepsilon^3\int_{-h}^{0} u_3\partial z' + O\left(\varepsilon^4\right).\end{aligned}$$

Therefore,

$$\overline{u} = \varepsilon u_1 + \varepsilon^2\left(\frac{1}{h}\int_{-h}^{0} u_2\partial z'\right) + \varepsilon^3\left(-\frac{\eta_1}{h^2}\int_{-h}^{0} u_2\partial z' + \frac{\eta_1}{h}u_2\big|_{z=0} + \frac{1}{h}\int_{-h}^{0} u_3\partial z'\right) + O\left(\varepsilon^4\right)$$

and depth-averaged flow variables can be defined to correspond with each order of $\overline{u}$:

$$\begin{aligned} \overline{u_1} &= u_1 \\ \overline{u_2} &= \frac{1}{h}\int_{-h}^{0} u_2 \partial z' \\ \overline{u_3} &= \frac{1}{h}\int_{-h}^{0} u_3 \partial z' - \frac{\eta_1 \overline{u_2}}{h} + \frac{\eta_1 u_2\big|_{z=0}}{h} \end{aligned} \tag{72}$$

Substituting (14) and (22) into (72) for u_2 and u_3, respectively, and integrating yields

$$\overline{u_2} = U_2 + \frac{h^2}{3}\frac{\partial^2 u_1}{\partial x^2} \tag{73}$$

and

$$\begin{aligned} \overline{u_3} = U_3 &+ \frac{h^4}{45}\frac{\partial^4 u_1}{\partial x'^4} + \frac{h^2}{3}\left(\frac{\partial^2 \overline{u_2}}{\partial x'^2} + \frac{\partial^2 v_1}{\partial x' dy'}\right) \\ &+ h\frac{\partial h}{\partial \underset{\sim}{x}}\frac{\partial u_1}{\partial x'} + \frac{h\eta_1}{3}\frac{\partial^2 u_1}{\partial x'^2} - \frac{h\eta_1}{3}\frac{\partial^2 u_1}{\partial x^2}. \end{aligned} \tag{74}$$

Similarly it can be shown that the depth-averaged alongshore flow variables are given by

$$\overline{v_1} = v_1, \tag{75}$$

$$\overline{v_2} = V_2 + \frac{h^2}{3}\frac{\partial^2 u_1}{\partial x \partial y}, \tag{76}$$

and

$$\begin{aligned} \overline{v_3} = V_3 &+ \frac{h^4}{45}\frac{\partial^4 u_1}{\partial x'^3 \partial y'} + \frac{h^2}{3}\left(\frac{\partial^2 \overline{u_2}}{\partial x' \partial y'} + \frac{\partial^2 v_1}{dy'^2}\right) \\ &+ \frac{h}{2}\left(\frac{\partial h}{\partial \underset{\sim}{x}}\frac{\partial u_1}{\partial y'} + \frac{\partial h}{\partial \underset{\sim}{y}}\frac{\partial u_1}{\partial x'}\right) + \frac{h\eta_1}{3}\frac{\partial^2 u_1}{\partial x'^2} - \frac{h\eta_1}{3}\frac{\partial^2 u_1}{\partial x' \partial y'}. \end{aligned} \tag{77}$$

APPENDIX B

Consider the set of velocity potential functions $\left\{\phi_i : u_i = \frac{\partial \phi_i}{\partial x}, v_i = \frac{\partial \phi_i}{\partial y}\right\}$. To make necessary substitutions to the depth-averaged governing equations, these functions must be expressed in a depth-averaged form. For convenience, define $\overline{\phi_i}$ to be the average value of ϕ_i from the bottom to the still sea surface

$$\overline{\phi_i} \equiv \frac{1}{h}\int_{-h}^{0} \phi_i dz. \tag{78}$$

Since u_1 and v_1 are independent of z, so is ϕ_1. Applying Leibniz's Integration Theorem,

$$\overline{u_2} = \frac{1}{h}\int_{-h}^{0} \frac{\partial \phi_2}{\partial x} dz = \frac{\partial \overline{\phi_2}}{\partial x} + \varepsilon \frac{\left(\overline{\phi_2} - \phi_2\left(-h\right)\right)}{h}\frac{\partial h}{\partial \tilde{x}}. \tag{79}$$

To eliminate the explicit boundary value, $\phi_2\left(-h\right)$, from equation (79), recall Bernoulli's equation

$$p = -\frac{\partial \phi}{\partial t} - \frac{u^2 + v^2 + w^2}{2} - z. \tag{80}$$

Substituting the perturbation expansions (4b-e) and the definitions of ϕ_i, and collecting terms of like order, we obtain

$$\eta_1 = -\frac{\partial \phi_1}{\partial t}, \tag{81}$$

$$\eta_2 + \left(\frac{z^2}{2} + hz\right)\frac{\partial^2 u_1}{\partial x \partial t} = -\frac{\partial \phi_2}{\partial t} - \frac{u_1^2}{2}. \tag{82}$$

Hence, using equation (78),

$$\frac{\partial \overline{\phi_2}}{\partial t} = -\eta_2 + \frac{h^2}{3}\frac{\partial^2 u_1}{\partial x \partial t} - \frac{u_1^2}{2} \tag{83}$$

and

$$\frac{\partial \phi_2(-h)}{\partial t} = -\eta_2 + \frac{h^2}{2}\frac{\partial^2 u_1}{\partial x \partial t} - \frac{u_1^2}{2}, \tag{84}$$

so that

$$\frac{\partial}{\partial t}\left(\overline{\phi_2} - \phi_2(-h)\right) = -\frac{h^2}{6}\frac{\partial^2 u_1}{\partial x \partial t}. \tag{85}$$

Integrating in time, setting the arbitrary integration constant to zero,

$$\overline{\phi_2} - \phi_2(-h) = -\frac{h^2}{6}\frac{\partial u_1}{\partial x}. \tag{86}$$

Thus, equation (79) simplifies to

$$\overline{u_2} = \frac{\partial \overline{\phi_2}}{\partial x} - \varepsilon \frac{h}{6}\frac{\partial u_1}{\partial x}\frac{\partial h}{\partial \tilde{x}}. \tag{87}$$

Similarly it can be shown that

$$\overline{v_2} = \frac{\partial \overline{\phi_2}}{\partial y} - \varepsilon \frac{h}{6}\frac{\partial u_1}{\partial x}\frac{\partial h}{\partial \tilde{y}}. \tag{88}$$

Applying the same procedure to the next higher order yields

$$\overline{u_3} = \frac{\partial \overline{\phi_3}}{\partial x} - \frac{\eta_1 h}{3}\frac{\partial^2 u_1}{\partial x^2} + O(\varepsilon), \tag{89}$$

$$\overline{v_3} = \frac{\partial \overline{\phi_3}}{\partial y} - \frac{\eta_1 h}{3}\frac{\partial^2 u_1}{\partial x \partial y} + O(\varepsilon). \tag{90}$$

APPENDIX C

Substitution of the assumed waveforms for ϕ_1 (equation (44)) and η_1 (equation (45)) and the solution for η_2 $\left(\overline{\phi_2}=0\right)$ derived in Appendix D (equation (95)) into the governing equation (43) yields the following expressions for each of the terms:

$O\left(1\right)$ terms:

$$\begin{aligned}&\frac{\partial^2\phi_1}{\partial t^2}-h\frac{\partial^2\phi_1}{\partial x^2}\\&=\sum_p\sum_q\left[\varepsilon\left(-2R_{1p,q}h^{1/2}+\frac{A_{p,q}}{2h^{1/2}}\frac{\partial h}{\partial\tilde{x}}\right)+\varepsilon^2\left(\frac{ih}{\omega_p}\frac{\partial R_{1p,q}}{\partial\tilde{x}}-2R_2h^{1/2}\right)+O\left(\varepsilon^3\right)\right]\exp\left[i\varphi_{p,q}\right].\end{aligned}\tag{91}$$

$O\left(\varepsilon\right)$ terms:

$$-h\frac{\partial^2\phi_1}{\partial y^2}=\sum_p\sum_q\left(-\frac{ihl_q^2A_{p,q}}{\omega_p}+\varepsilon\left(\frac{-2hS_{1p,q}l_q}{\omega_p}\right)+O(\varepsilon^2)\right)\exp\left[i\varphi_{p,q}\right],$$

$$-\frac{h^2}{3}\frac{\partial^4\phi_1}{\partial x^2\partial t^2}=\sum_p\sum_q\left(\frac{ih\omega_p^3A_{p,q}}{3}+\varepsilon\left(\frac{2h^{3/2}\omega_p^2R_{1p,q}}{3}-\frac{h^{1/2}\omega_p^2A_{p,q}}{6}\frac{\partial h}{\partial\tilde{x}}\right)+O\left(\varepsilon^2\right)\right)\exp\left[i\varphi_{p,q}\right],$$

$$-\frac{\partial h}{\partial\tilde{x}}\frac{\partial\phi_1}{\partial x}=\sum_p\sum_q\left(\frac{-A_{p,q}}{h^{1/2}}\frac{\partial h}{\partial\tilde{x}}+\varepsilon\left(\frac{iR_{1p,q}}{\omega_p}\frac{\partial h}{\partial\tilde{x}}\right)+O\left(\varepsilon^2\right)\right)\exp\left[i\varphi_{p,q}\right],$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial x}\right)^2=\sum_p\sum_q\sum_r\sum_s\left(\frac{-i\omega_p}{2h}A_{r,s}-\varepsilon\frac{\omega_pR_{1r,s}}{h^{1/2}\omega_r}+O\left(\varepsilon^2\right)\right)A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right],$$

$$\begin{aligned}-\frac{\partial}{\partial x}\left(\eta_1\frac{\partial\phi_1}{\partial x}\right)=\sum_p\sum_q\sum_r\sum_s&\left(\frac{-i\omega_p}{h}A_{r,s}+\varepsilon\left(\frac{A_{r,s}}{2h^{3/2}}\frac{\partial h}{\partial\tilde{x}}-\frac{\omega_pR_{1r,s}}{h^{1/2}\omega_r}-\frac{2R_{1r,s}}{h^{1/2}}\right)+O\left(\varepsilon^2\right)\right)\\&\times A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right].\end{aligned}$$

$O\left(\varepsilon^2\right)$ terms:

$$-\frac{h^2}{3}\frac{\partial^4\phi_1}{\partial y^2\partial t^2}=\sum_p\sum_q\frac{h^2\omega_p l_q^2 A_{p,q}}{3}\exp\left[i\varphi_{p,q}\right],$$

$$-\frac{h^4}{45}\frac{\partial^6\phi_1}{\partial x^4\partial t^2}=\sum_p\sum_q\frac{-ih^2\omega_p^5}{45}A_{p,q}\exp\left[i\varphi_{p,q}\right],$$

$$-\frac{\partial h}{\partial\tilde{y}}\frac{\partial\phi_1}{\partial y}=\sum_p\sum_q\left(-\frac{\partial h}{\partial\tilde{y}}\frac{l_q}{\omega_p}\right)A_{p,q}\exp\left[i\varphi_{p,q}\right],$$

$$-h\frac{\partial h}{\partial\tilde{x}}\frac{\partial^3\phi_1}{\partial x\partial t^2}=\sum_p\sum_q h^{1/2}\frac{\partial h}{\partial\tilde{x}}\omega_p^2 A_{p,q}\exp\left[i\varphi_{p,q}\right],$$

$$-h\frac{\partial}{\partial t}\left(\eta_1\frac{\partial^3\phi_1}{\partial x^2\partial t}\right)=\sum_p\sum_q\sum_r\sum_s i\omega_p\omega_r^2 A_{r,s}A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right],$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial y}\right)^2=\sum_p\sum_q\sum_r\sum_s\frac{-il_s^2}{\omega_r}A_{r,s}A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right],$$

$$\frac{h^2}{2}\frac{\partial}{\partial t}\left(\frac{\partial^2\phi_1}{\partial x^2}\right)^2=\sum_p\sum_q\sum_r\sum_s i\omega_p\omega_r^2 A_{r,s}A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right],$$

$$-\frac{h^2}{3}\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial x}\frac{\partial^3\phi_1}{\partial x^3}\right)=\sum_p\sum_q\sum_r\sum_s\frac{-i\omega_p\omega_r^2}{3}A_{r,s}A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right],$$

$$\begin{aligned}-\frac{\partial}{\partial x}\left(\eta_2\frac{\partial\phi_1}{\partial x}\right)=\sum_p\sum_q\sum_r\sum_s\Bigg[&\left(\frac{-\omega_p}{\omega_r}\left(\frac{A_{r,s}}{2h^{3/2}}\frac{\partial h}{\partial\tilde{x}}+\frac{2R_{1r,s}}{h^{1/2}}\right)+i\frac{\omega_p l_s^2}{\omega_r^2}A_{r,s}\right)A_{p-r,q-s}\\&+\sum_m\sum_n\frac{i\omega_p}{h^2}A_{m,n}A_{r,s}A_{p-r-m,q-s-n}\Bigg]\exp\left[i\varphi_{p,q}\right],\end{aligned}$$

$$-\frac{\partial}{\partial y}\left(\eta_1\frac{\partial\phi_1}{\partial y}\right)=\sum_p\sum_q\sum_r\sum_s\frac{-il_s l_q}{\omega_r}A_{r,s}A_{p-r,q-s}\exp\left[i\varphi_{p,q}\right].$$

APPENDIX D

To determine η_2, consider the surface boundary condition

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}, \qquad z = \eta. \tag{92}$$

Substituting the perturbation expansions (4a-d) and scaling relations (5) into equation (92) and collecting terms of like order yields

$$\left(\frac{\partial \eta_1}{\partial t'} + h\frac{\partial u_1}{\partial x'}\right) + \varepsilon\left(\frac{\partial \eta_2}{\partial t'} + h\left(\frac{\partial \overline{u_2}}{\partial x'} + \frac{\partial v_1}{\partial y'}\right) + u_1\frac{\partial h}{\partial \tilde{x}} + \frac{\partial\left(\eta_1 u_1\right)}{\partial x'}\right) = O(\varepsilon^2). \tag{93}$$

Now, substitute the assumed lowest order wavefield (44) and (45) (dropping the primes to simplify the notation) to express each term involving η_1, u_1, and v_1 in terms of the spectral amplitudes $A_{p,q}$:

$$\frac{\partial \eta_1}{\partial t} = \sum_p \sum_q -i\omega_p A_{p,q} \exp\left[i\varphi_{p,q}\right],$$

$$h\frac{\partial u_1}{\partial x} = h\frac{\partial^2 \phi_1}{\partial x^2} = \sum_p \sum_q \left[i\omega_p A_{p,q} + \varepsilon\left(2R_{1p,q}h^{1/2} - \frac{A_{p,q}}{2h^{1/2}}\frac{\partial h}{\partial \tilde{x}}\right) + O\left(\varepsilon^2\right)\right]\exp\left[i\varphi_{p,q}\right],$$

$$h\left(\frac{\partial \overline{u_2}}{\partial x} + \frac{\partial v_1}{\partial y}\right) = h\left(\frac{\partial^2 \overline{\phi_2}}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}\right) + O\left(\varepsilon\right) = \sum_p \sum_q i\frac{hl_q^2 A_{p,q}}{\omega_p}\exp\left[i\varphi_{p,q}\right],$$

$$u_1\frac{\partial h}{\partial \tilde{x}} = \frac{\partial h}{\partial \tilde{x}}\frac{\partial \phi_1}{\partial x} = \sum_p \sum_q \frac{1}{h^{1/2}}\frac{\partial h}{\partial \tilde{x}} A_{p,q}\exp\left[i\varphi_{p,q}\right],$$

$$\frac{\partial(\eta_1 u_1)}{\partial x} = \frac{\partial}{\partial x}\left(\eta_1 \frac{\partial \phi_1}{\partial x}\right) = \frac{\partial}{\partial x}\left(\left(\sum_p \sum_q A_{p,q} \exp\left[i\varphi_{p,q}\right]\right)\left(\sum_r \sum_s \frac{1}{h^{1/2}} A_{r,s} \exp\left[i\varphi_{r,s}\right]\right)\right)$$

$$= \sum_p \sum_q \sum_r \sum_s \frac{i\omega_p A_{r,s} A_{p-r,q-s}}{h} \exp\left[i\varphi_{p,q}\right] + O(\varepsilon).$$

Hence $\frac{\partial \eta_2}{\partial t}$ is given by,

$$\frac{\partial \eta_2}{\partial t} = \sum_p \sum_q \left(-\frac{A_{p,q}}{2h^{1/2}} \frac{\partial h}{\partial \tilde{x}} - 2R_{1p,q} h^{1/2} - i\left(\frac{h l_q^2 A_{p,q}}{\omega_p} + \sum_r \sum_s \frac{\omega_p A_{r,s} A_{p-r,q-s}}{h}\right)\right) \exp\left[i\varphi_{p,q}\right] + O(\varepsilon). \tag{94}$$

Finally, integrating equation (94) with respect to time and setting the arbitrary integration constant to zero,

$$\eta_2 = \sum_p \sum_q \left(\frac{h l_q^2 A_{p,q}}{\omega_p} + \sum_r \sum_s \frac{A_{r,s} A_{p-r,q-s}}{h} + i\left(-\frac{A_{p,q}}{2h^{1/2}\omega_p} \frac{\partial h}{\partial \tilde{x}} - \frac{2R_{1p,q} h^{1/2}}{\omega_p}\right)\right) \exp\left[i\varphi_{p,q}\right] + O(\varepsilon). \tag{95}$$

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